QUANTUM MECHANICS I (523)
PROBLEM SET 9 (hand in November 12)

33) For free-particle states on requires the normalization

$$\langle E\ell m|E'\ell' m'\rangle = \delta(E-E')\delta_{\ell,\ell'}\delta_{m,m'}.$$

The wave function can be written as the radial wave function times the appropriate spherical harmonic

$$\langle r\theta\phi|E\ell m\rangle = R_{E\ell}(r)Y_{\ell m}(\theta,\phi) = c_{E,\ell}\; j_\ell(kr)Y_{\ell m}(\theta,\phi),$$

where the spherical Bessel function provides the relevant solution for the radial part. Determine the normalization constant $c_{E,\ell}$ in more detail than what is done in the book.

34) A particle in a spherically symmetrical potential is known to be in an eigenstate of $\boldsymbol{\ell}^2$ and ℓ_z with eigenvalues $\hbar^2\ell(\ell+1)$ and m, respectively. Prove that the following expectation values w.r.t this state are satisfied:

$$\langle \ell_x\rangle = \langle \ell_y\rangle = 0$$

and

$$\langle \ell_x^2\rangle = \langle \ell_y^2\rangle = \frac{[\ell(\ell+1)\hbar^2 - m^2\hbar^2]}{2}.$$

Try to interpret this result.

35) Suppose a half-integer ℓ-value, say 1/2, were allowed for orbital angular momentum. From

$$\ell_+ Y_{1/2,1/2}(\theta,\phi) = 0,$$

one may deduce

$$Y_{1/2,1/2}(\theta,\phi) \propto \exp\{i\phi/2\}\sqrt{\sin\theta}.$$

Try to construct $Y_{1/2,-1/2}$ by

a) applying ℓ_- to $Y_{1/2,1/2}$ and

b) using

$$\ell_- Y_{1/2,-1/2}(\theta, \phi) = 0.$$

Show that these two procedures lead to contradictory results (lending support to the notion that half-integer ℓ-values are not possible).

36) Calculate the following commutation relations:

a)

$$[\ell_i, x_j]$$

b)

$$[\ell_i, p_j]$$

c)

$$\left[p_i, \frac{1}{r}\right]$$

d)

$$\left[p_i, \frac{x_j}{r}\right]$$

e)

$$[(\boldsymbol{\ell} \times \boldsymbol{p})_i, p_j]$$

f)

$$\left[(\boldsymbol{\ell} \times \boldsymbol{p})_i, \frac{1}{r}\right],$$

where i and j correspond to x, y or z, as usual.