QUANTUM MECHANICS (471)
PROBLEM SET 4 (hand in September 30)

12) (10 points) Consider the operator $x$ in the momentum representation (in one dimension). Prove

$$\langle p'| \, x \, |\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle,$$

and

$$\langle \beta| \, x \, |\alpha\rangle = i\hbar \int dp' \; \phi^*_\beta(p') \frac{\partial}{\partial p'} \phi_\alpha(p'),$$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum space wave functions.

13) (10 points) Consider an electron in a static, uniform magnetic field with strength $B$ pointing in the $z$-direction. At $t = 0$ the electron is in an eigenstate of $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$ with eigenvalue $\hbar/2$, where $\hat{\boldsymbol{n}}$ is a unit vector, in the $yz$-plane making an angle $\beta$ with the $z$-axis.

a) Determine the probability to obtain $\hbar/2$ for a measurement of $S_y$ as a function of time.

b) Calculate the expectation value of $S_y$ as a function of time and make sure that the result makes sense for both $\beta = 0$ and $\pi/2$.

14) (10 points) A two-state system has a Hamiltonian given by

$$H = h_{11} |1\rangle \langle 1| + h_{22} |2\rangle \langle 2| + h_{12} \left( |1\rangle \langle 2| + |2\rangle \langle 1| \right),$$

where $h_{11}, h_{22}$, and $h_{12}$ are real numbers, and $|1\rangle$ and $|2\rangle$ are eigenkets of some other observable different from $H$.

a) Find the energy eigenkets and corresponding eigenvalues. Check that the limit $h_{12} \to 0$ makes sense for your results.

b) Suppose by mistake the Hamiltonian was written as

$$H = h_{11} |1\rangle \langle 1| + h_{22} |2\rangle \langle 2| + h_{12} |1\rangle \langle 2| .$$

What problem will occur when time evolution is considered with this illegal Hamiltonian? Illustrate this problem explicitly by considering the evolution of some general initial state $|\psi(t_0)\rangle = \lambda_1 |1\rangle + \lambda_2 |2\rangle$ while assuming $h_{11} = h_{22} = 0$ for simplicity.

It is helpful not to use the results of part a) while solving part b)!

15) (10 points) Consider a spinless particle in one dimension with a Hamiltonian given by

$$H = \frac{p^2}{2m} + V(x).$$

a) Evaluate the commutator

$$A = [H, x].$$

Next evaluate

$$[A, x].$$

b) Use this result and appropriate completeness relations to show that

$$\sum_i |\langle E_j| x |E_i\rangle|^2 (E_i - E_j) = \frac{\hbar^2}{2m},$$

where the $\{|E_i\rangle\}$ are energy eigenkets with corresponding eigenvalues $E_i$.