Bosons with interactions

- Fermi sea for fermions because of maximum occupation by one for each sp state
- Bosons can occupy sp states in any number
- Noninteracting ground state: all bosons in lowest sp state
- Macroscopic occupation: BEC
- Study in the presence of nonnegligible interaction
- Boson Hamiltonian $\hat{H} = \hat{T} + \hat{V}$
- with $$\hat{T} = \sum_{\alpha\beta} \langle\alpha| T |\beta\rangle a^\dagger_\alpha a_\beta$$
- and tp interaction $$\hat{V} = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle\alpha\beta| V |\gamma\delta\rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma$$
- with $$\langle\alpha\beta| V |\gamma\delta\rangle = (\alpha\beta| V |\gamma\delta) + (\alpha\beta| V |\delta\gamma)$$

Boson single-particle propagator

- Definition $i\hbar G(\alpha,\beta;t,t') = \langle \Psi_0^N | \mathcal{T}[a_{\alpha_H}(t) a^\dagger_{\beta_H}(t')] | \Psi_0^N \rangle$
- No sign involved with time ordering so

$$
\begin{aligned}
i\hbar G(\alpha,\beta;\tau) &= \langle \Psi_0^N | \theta(\tau) a_\alpha \mathrm{e}^{-\frac{i}{\hbar}(\hat{H}-E_0^N)\tau} a^\dagger_\beta + \theta(-\tau) a^\dagger_\beta \mathrm{e}^{\frac{i}{\hbar}(\hat{H}-E_0^N)\tau} a_\alpha | \Psi_0^N \rangle \\
&= \theta(\tau) \sum_m \mathrm{e}^{-\frac{i}{\hbar}(E_m^{N+1}-E_0^N)\tau} \langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a^\dagger_\beta | \Psi_0^N \rangle \\
&+ \theta(-\tau) \sum_n \mathrm{e}^{\frac{i}{\hbar}(E_n^{N-1}-E_0^N)\tau} \langle \Psi_0^N | a^\dagger_\beta | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle
\end{aligned}
$$

- Energy formulation

$$
\begin{aligned}
G(\alpha,\beta;E) &= \int_{-\infty}^{+\infty} d\tau \, \mathrm{e}^{\frac{i}{\hbar}E\tau} G(\alpha,\beta;\tau) \\
&= \langle \Psi_0^N | a_\alpha \frac{1}{E-\hat{H}+E_0^N+i\eta} a^\dagger_\beta - a^\dagger_\beta \frac{1}{E+\hat{H}-E_0^N-i\eta} a_\alpha | \Psi_0^N \rangle \\
&= \sum_m \frac{\langle \Psi_0^N | a_\alpha | \Psi_m^{N+1} \rangle \langle \Psi_m^{N+1} | a^\dagger_\beta | \Psi_0^N \rangle}{E-(E_m^{N+1}-E_0^N)+i\eta} - \sum_n \frac{\langle \Psi_0^N | a^\dagger_\beta | \Psi_n^{N-1} \rangle \langle \Psi_n^{N-1} | a_\alpha | \Psi_0^N \rangle}{E+(E_n^{N-1}-E_0^N)-i\eta}
\end{aligned}
$$

Noninteracting boson propagator

- Noninteracting ground state for $\hat{T} = \sum_{\alpha} \varepsilon_\alpha a_\alpha^\dagger a_\alpha$ or $\hat{H}_0$

$$|\Phi_0^N\rangle = \frac{1}{\sqrt{N!}} \left(a_0^\dagger\right)^N |0\rangle$$ with lowest sp energy ε_0

- Use

$$\begin{aligned} a_0^\dagger \left|\Phi_0^N\right\rangle &= \sqrt{N+1}\left|\Phi_0^{N+1}\right\rangle \\ a_0 \left|\Phi_0^N\right\rangle &= \sqrt{N}\left|\Phi_0^{N-1}\right\rangle \end{aligned}$$

$\alpha \neq 0$

$$\begin{aligned} a_\alpha^\dagger \left|\Phi_0^N\right\rangle &= \left|\alpha(0)^N\right\rangle \\ a_\alpha \left|\Phi_0^N\right\rangle &= 0 \end{aligned}$$

- Noninteracting propagator

$$\begin{aligned} i\hbar G^{(0)}(\alpha,\beta;\tau) &= \delta_{\alpha\beta} \mathrm{e}^{-\frac{i}{\hbar}\varepsilon_\alpha \tau} \left\{\theta(\tau)(N\delta_{\alpha,0}+1) + \theta(-\tau) N\delta_{\alpha,0}\right\} \\ &= \delta_{\alpha\beta} \mathrm{e}^{-\frac{i}{\hbar}\varepsilon_\alpha \tau} \left\{\theta(\tau) + N\delta_{\alpha,0}\right\} \end{aligned}$$

- Noncondensate only forward while condensate both
- Ground-state energy $E_0^N = \langle\Phi_0^N|\,\hat{T}\,|\Phi_0^N\rangle = N\varepsilon_0$
- Chemical potential $\mu = E_0^N - E_0^{N-1} = E_0^{N+1} - E_0^N = \varepsilon_0$

Equations of motion

- Same procedure as for fermions leads to

$$i\hbar\frac{\partial}{\partial t}G(\alpha,\beta;t-t') = \delta(t-t')\delta_{\alpha,\beta} + \langle\Psi_0^N|\,\theta(t-t')\frac{\partial a_{\alpha_H}(t)}{\partial t}a^\dagger_{\beta_H}(t') + \theta(t'-t)a^\dagger_{\beta_H}(t')\frac{\partial a_{\alpha_H}(t)}{\partial t}\,|\Psi_0^N\rangle$$

- with first term from

$$\frac{d}{dt}\theta(t-t') = \delta(t-t') = -\frac{d}{dt}\theta(t'-t)$$

- and $$\langle\Psi_0^N|\,a_{\alpha_H}(t)a^\dagger_{\beta_H}(t) - a^\dagger_{\beta_H}(t)a_{\alpha_H}(t))\,|\Psi_0^N\rangle = \delta_{\alpha,\beta}$$

- Equation of motion of boson removal operator in Heisenberg picture

$$i\hbar\frac{\partial a_{\alpha_H}(t)}{\partial t} = [a_{\alpha_H}(t),\hat{H}]$$

$$= \sum_\gamma \langle\alpha|\,T\,|\gamma\rangle\, a_{\gamma_H}(t) + \frac{1}{2}\sum_{\gamma\delta\epsilon}\langle\alpha\gamma|\,V\,|\delta\epsilon\rangle\, a^\dagger_{\gamma_H}(t)a_{\epsilon_H}(t)a_{\delta_H}(t)$$

- Substitute

$$i\hbar\frac{\partial}{\partial t}G(\alpha,\beta;t-t') = \delta(t-t')\delta_{\alpha,\beta} + \sum_\gamma\langle\alpha|\,T\,|\gamma\rangle\,G(\gamma,\beta;t-t') + P_2$$

Tp propagator

- term with two-body interaction

$$P_2 = \frac{1}{2i\hbar}\sum_{\gamma\delta\epsilon}\langle\alpha\gamma|V|\delta\epsilon\rangle\left\{\theta(t-t')a^\dagger_{\gamma_H}(t)a_{\epsilon_H}(t)a_{\delta_H}(t)a^\dagger_{\beta_H}(t')\right.$$
$$\left.+\theta(t'-t)a^\dagger_{\beta_H}(t')a^\dagger_{\gamma_H}(t)a_{\epsilon_H}(t)a_{\delta_H}(t)\right\}$$

- Involves tp propagator
- In general form

$$i\hbar G_{II}(\alpha t_\alpha,\beta t_\beta,\gamma t_\gamma,\delta t_\delta)=\langle\Psi_0^N|\mathcal{T}[a_{\beta_H}(t_\beta)a_{\alpha_H}(t_\alpha)a^\dagger_{\gamma_H}(t_\gamma)a^\dagger_{\delta_H}(t_\delta)]|\Psi_0^N\rangle$$

- So sp propagator reads

$$i\hbar\frac{\partial}{\partial t}G(\alpha,\beta;t-t')=\delta(t-t')\delta_{\alpha,\beta}+\sum_\gamma\langle\alpha|T|\gamma\rangle G(\gamma,\beta;t-t')$$
$$+\frac{1}{2}\sum_{\gamma\delta\epsilon}\langle\alpha\gamma|V|\delta\epsilon\rangle G_{II}(\delta t,\epsilon t,\beta t',\gamma t^+)$$

- As for fermions: first step in hierarchy

Perturbation expansion and problem with Wick

- So far complete equivalence with fermion problem
- But: direct application of Wick's theorem not possible
- Wick: requires normal ordering that leads to vanishing expectation values in noninteracting ground state
- But $a_0 \left|\Phi_0^N\right\rangle = \sqrt{N} \left|\Phi_0^{N-1}\right\rangle , \qquad a_0^\dagger \left|\Phi_0^N\right\rangle = \sqrt{N+1} \left|\Phi_0^{N+1}\right\rangle$
- shows that this is not possible here for "0" orbital
- Condensate orbital requires special treatment
- Genuine boson perturbation theory: later
- For now: quickly generate boson mean-field equation
- Convert to equivalent fermion problem

Equivalent fermion problem

Hydrogen molecule with two-electron ground state with total S=0 can be factorized as

$$\Psi(\boldsymbol{r}_1 m_{s_1}, \boldsymbol{r}_2 m_{s_2}) = \Phi(\boldsymbol{r}_1, \boldsymbol{r}_2)\Xi(m_{s_1}, m_{s_2})$$

- with spin part $\Xi(m_{s_1}, m_{s_2}) = \frac{1}{\sqrt{2}}\left(\delta_{m_{s_1},+\frac{1}{2}}\delta_{m_{s_2},-\frac{1}{2}} - \delta_{m_{s_1},-\frac{1}{2}}\delta_{m_{s_2},+\frac{1}{2}}\right)$
- carrying fermion antisymmetry
- Consequently spatial wave function must be symmetric!
- Requires a wave function for two spinless bosons

- Generalize to an arbitrary number of particles
- Convert N interacting bosons to a fermion problem by adding (fictitious) quantum number λ that can take on N values

Equivalent N fermion problem

- Each sp boson state α is now N-fold degenerate and generates N fermion sp states $(\alpha\lambda_\alpha)$ with $\lambda_\alpha = 1, \ldots, N$
- Take fermion Hamiltonian diagonal in fictitious quantum number

$$\langle \alpha\lambda_\alpha | T | \beta\lambda_\beta \rangle = \langle \alpha | T | \beta \rangle \, \delta_{\lambda_\alpha, \lambda_\beta}$$

- and $$(\alpha\lambda_\alpha, \beta\lambda_\beta | V | \gamma\lambda_\gamma, \delta\lambda_\delta) = (\alpha\beta | V | \gamma\delta) \, \delta_{\lambda_\alpha \lambda_\gamma} \delta_{\lambda_\beta \lambda_\delta}$$
- Noninteracting ground state N fermions: closed shell $|\Phi_0^N\rangle = \prod_{\lambda=1}^{N} a_{0\lambda}^\dagger |0\rangle$
- all referring to lowest sp boson state
- Ground-state: singlet in the fictitious quantum number and factorizes $$\langle \alpha_1\lambda_1, \ldots, \alpha_N\lambda_N | \Phi_0^N \rangle = \left[\prod_{i=1}^{N} \delta_{\alpha_i, 0} \right] \frac{1}{\sqrt{N!}} \mathrm{Det}[\delta_{i,\lambda_j}]_{i,j=1,\ldots,N}$$
- Symmetric part: noninteracting boson ground state and antisymmetric Slater determinant representing closed-shell in λ
- Interaction maintains singlet character: factorization remains

Fictitious fermion problem & Hartree-Bose

- Perturbation and diagrams from before but include λ summation
- Since λ is conserved (diagonal in the Hamiltonian) at each vertex, generates a factor N for each closed fermion loop
- In addition, the summation over occupied fermion states only refers to the condensate orbital

Hartree-Bose

- Consider HF self-energy $\Sigma^{HF}(\gamma,\delta) = \sum_{h=1}^{N} \left[(\gamma h| V |\delta h) - (\gamma h| V |h\delta) \right]$
- with summation over occupied sp (hole) state $|h\rangle = \sum_{\mu} z_{\mu}^{h} |\mu\rangle$
- Boson counterpart: introduce fictitious quantum and restrict hole summation to condensate orbital c

Hartree-Bose

- So HB self-energy

$$\Sigma^{HB}(\gamma\lambda_\gamma,\delta\lambda_\delta)=\sum_{\lambda_c}\left[(\gamma\lambda_\gamma,c\lambda_c|\,V\,|\delta\lambda_\delta,c\lambda_c)-(\gamma\lambda_\gamma,c\lambda_c|\,V\,|c\lambda_c,\delta\lambda_\delta)\right]$$

- Use $(\gamma\lambda_\gamma\ c\lambda_c|\,V\,|\delta\lambda_\delta\ c\lambda_c)=(\gamma c|\,V\,|\delta c)\,\delta_{\lambda_\gamma\lambda_\delta}$

$$(\gamma\lambda_\gamma\ c\lambda_c|\,V\,|c\lambda_c\ \delta\lambda_\delta)=(\gamma c|\,V\,|c\delta)\,\delta_{\lambda_\gamma\lambda_c}\delta_{\lambda_\delta\lambda_c}$$

- so $\Sigma^{HB}(\gamma\lambda_\gamma,\delta\lambda_\delta)=\Sigma^{HB}(\gamma,\delta)\delta_{\lambda_\gamma,\lambda_\delta}$
- and summing over λ_c yields $\Sigma^{HB}(\gamma,\delta)=N\,(\gamma c|\,V\,|\delta c)-(\gamma c|\,V\,|c\delta)$
- consistent with diagram rules discussed above
- Not-yet determined condensate orbital $|c\rangle=\sum_\mu z^c_\mu\,|\mu\rangle$ from lowest solution of

$$\sum_\delta\left\{\langle\gamma|\,T\,|\delta\rangle+\Sigma^{HB}(\gamma,\delta)\right\}z^c_\delta=\varepsilon_c z^c_\gamma$$

more HB

- Like HF, HB is self-consistent so HB mean field is (general basis)

$$\Sigma^{HB}(\gamma,\delta) = \sum_{\mu\nu} \{N\,(\gamma\mu|\,V\,|\delta\nu) - (\gamma\mu|\,V\,|\nu\delta)\}\, z_\mu^{c*} z_\nu^c$$

- the interaction averaged over the condensate density (itself determining the condensate orbital with $\sum_\mu |z_\mu^c|^2 = 1$)
- Rewrite $\sum_\delta \Sigma^{HB}(\gamma,\delta) z_\delta^c$

$$\begin{aligned} \sum_\delta \Sigma^{HB}(\gamma,\delta) z_\delta^c &= \sum_{\mu\nu\delta} \{N\,(\gamma\mu|\,V\,|\delta\nu) - (\gamma\mu|\,V\,|\nu\delta)\}\, z_\mu^{c*} z_\nu^c z_\delta^c \\ &= \sum_{\mu\nu\delta} N\,(\gamma\mu|\,V\,|\delta\nu)\, z_\mu^{c*} z_\nu^c \times\ z_\delta^c - \sum_{\mu\delta\nu} (\gamma\mu|\,V\,|\delta\nu)\, z_\mu^{c*} z_\nu^c \times\ z_\delta^c \\ &= \sum_\delta W_{HB}(\gamma,\delta) z_\delta^c \end{aligned}$$

- with

$$W_{HB}(\gamma,\delta) = (N-1) \sum_{\mu\nu} (\gamma\mu|\,V\,|\delta\nu)\, z_\mu^{c*} z_\nu^c$$

- So an equivalent form of the HB eigenvalue equation is

$$\sum_\delta \{\langle\gamma|\,T\,|\delta\rangle + W_{HB}(\gamma,\delta)\}\, z_\delta^c = \varepsilon_c z_\gamma^c$$

HB ground-state energy

- Use HF results in two different forms

$$(\text{HF:}) \quad E_0^N = \sum_h \langle h|\, T\, |h\rangle + \frac{1}{2}\sum_{h_1,h_2} \left[(h_1 h_2|\, V\, |h_1 h_2) - (h_1 h_2|\, V\, |h_2 h_1) \right]$$

$$(\text{HF:}) \quad E_0^N = \frac{1}{2}\sum_h \left[\langle h|\, T\, |h\rangle + \varepsilon_h \right]$$

- Restricting sum to condensate orbital and including fake quantum numbers yields

$$(\text{HB:}) \quad E_0^N = \sum_{\lambda_c} \langle c\lambda_c|\, T\, |c\lambda_c\rangle + \frac{1}{2}\sum_{\lambda_{c_1},\lambda_{c_2}} \left[(c\lambda_{c_1}, c\lambda_{c_2}|\, V\, |c\lambda_{c_1}, c\lambda_{c_2}) - (c\lambda_{c_1}, c\lambda_{c_2}|\, V\, |c\lambda_{c_2}, c\lambda_{c_1}) \right]$$

$$(\text{HB:}) \quad E_0^N = \frac{1}{2}\sum_{\lambda_c} \left[\langle c\lambda_c|\, T\, |c\lambda_c\rangle + \varepsilon_{c\lambda_c} \right]$$

- After summation

$$(\text{HB:}) \quad E_0^N = N \langle c|\, T\, |c\rangle + \frac{N(N-1)}{2} (cc|\, V\, |cc)$$

$$(\text{HB:}) \quad E_0^N = \frac{N}{2} \left(\langle c|\, T\, |c\rangle + \varepsilon_c \right)$$

HB in coordinate space

- Put bosons in local external potential (HO) $U(\boldsymbol{r})$ and employ

$$(\boldsymbol{r}_1\boldsymbol{r}_2|\,V\,|\boldsymbol{r}_3\boldsymbol{r}_4) = \delta(\boldsymbol{r}_1-\boldsymbol{r}_3)\delta(\boldsymbol{r}_2-\boldsymbol{r}_4)V(\boldsymbol{r}_1-\boldsymbol{r}_2)$$

- Replace $\alpha \equiv \boldsymbol{r}$ in HB equations and introduce wave function of condensate orbital $\phi_c(\boldsymbol{r}) \equiv z^c_{\boldsymbol{r}}$
- Yields $W_{HB}(\boldsymbol{r}) = (N-1)\int d\boldsymbol{r}'\; V(\boldsymbol{r}-\boldsymbol{r}')|\phi_c(\boldsymbol{r}')|^2$
- and HB equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{r}) + W_{HB}(\boldsymbol{r})\right]\phi_c(\boldsymbol{r}) = \varepsilon_c\phi_c(\boldsymbol{r})$$

- Ground state

$$\left\langle \boldsymbol{r}_1,\boldsymbol{r}_2,..,\boldsymbol{r}_N|\Phi_0^N\right\rangle = \phi_c(\boldsymbol{r}_1)\phi_c(\boldsymbol{r}_2)..\phi_c(\boldsymbol{r}_N)$$

- and condensate orbital minimizes ($\int d\boldsymbol{r}|\phi_c(\boldsymbol{r})|^2 = 1$) -- not shown

$$\frac{E_0^{HB}}{N} = \int d\boldsymbol{r}\phi_c^*(\boldsymbol{r})\left[-\frac{\hbar^2\nabla^2}{2m} + U(\boldsymbol{r}) + \frac{(N-1)}{2}\int d\boldsymbol{r}'\; V(\boldsymbol{r}-\boldsymbol{r}')|\phi_c(\boldsymbol{r}')|^2\right]\phi_c(\boldsymbol{r})$$

Gross-Pitaevskii (GP) equation for dilute systems

- HB can only be applied when the interaction is weak and nonsingular: not the case in the real world since atom-atom interaction is strongly repulsive when electron clouds overlap
- Mean-field not appropriate
- Real N-boson wave function will vanish whenever two atoms enter repulsive zone
- Beyond mean field: sum a particular set of diagrams as in free space (as for fermions) --> ladder diagrams
- Replace bare interaction by effective interaction in the medium
- For very dilute systems no medium effect so can use free $\mathcal{T}$
- At very low temperatures boson momenta very close to zero so only S-wave scattering contributes: characterized by one number
- Number is scattering length
- Simple way to incorporate is to replace bare interaction by so-called pseudopotential of zero range $V(\boldsymbol{r}-\boldsymbol{r}') \rightarrow g\delta(\boldsymbol{r}-\boldsymbol{r}')$
- Strength related to scattering length $g = \dfrac{4\pi\hbar^2 a}{m}$

Scattering theory

- Wave must asymptotically look like ($C_{\ell k}$ constant)

$$\psi_{\ell k}(r) \to C_{\ell k}[\cos\delta_{\ell k}\,\mathrm{j}_\ell(kr) - \sin\delta_{\ell k}\,\mathrm{n}_\ell(kr)]$$

- Free solutions spherical Bessel and Neumann functions with asymptotic behavior

$$\mathrm{j}_\ell(x) \to \frac{\sin(x - \ell\frac{\pi}{2})}{x}, \qquad \mathrm{n}_\ell(x) \to -\frac{\cos(x - \ell\frac{\pi}{2})}{x}$$

- Standard result from QM: low-energy limit of phase shift

$$\delta_{\ell k} \sim k^{2\ell+1}[1 + \mathcal{O}(k^2)]$$

- so only S-wave $\ell = 0$ survives
- Scattering length defined as $-\frac{1}{a} = \lim_{k\to 0} k\cot\delta_{0k}$ completely characterizes the (effective) interaction at low energy

Values for atomic collisions

- $a = 2.75$ nm for ^{23}Na
- $a = 5.77$ nm for ^{87}Rb
- $a = 1.45$ nm for ^{7}Li
- $a \sim -23$ nm for ^{85}Rb
- $a = 3.45$ nm for ^{133}Cs
- Positive scattering lengths lead to stable condensates
- Negative scattering lengths lead to collapse
- Tuning scattering length is sometimes possible by applying a magnetic field and exploiting atomic hyperfine structure
- Experimentally confirmed

$\mathcal{T}$-matrix

- Two-particle problem: replace m in all equations of Ch.6 by $m/2$
- Low-energy limit from integral equation for $\mathcal{T}$

$$\langle k'|\mathcal{T}^{\ell}(E)|k\rangle = \langle k'|V^{\ell}|k\rangle + \int_0^{\infty} dq\, q^2 \langle k'|V^{\ell}|q\rangle G^{(0)}(q;E)\langle q|\mathcal{T}^{\ell}(E)|k\rangle$$

- with propagator for free relative particle

$$G^{(0)}(q;E) = \frac{1}{E - \frac{\hbar^2 q^2}{m} + i\eta}$$

- Define half-on-shell $\mathcal{T}$-matrix (energy equals energy initial state)

$$\langle k'|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle = \langle k'|\,\mathcal{T}^{\ell}(E = \frac{\hbar^2 k^2}{m})\,|k\rangle$$

- Corresponding integral equation

$$\langle k'|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle = \langle k'|\,V^{\ell}\,|k\rangle + \frac{m}{\hbar^2}\int_0^{+\infty} dq\, q^2 \frac{\langle k'|\,V^{\ell}\,|q\rangle\,\langle q|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle}{k^2 - q^2 + i\eta}$$

- Diagonal element related to phase shift

$$\langle k|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle = -\frac{2\hbar^2}{m\pi k}\mathrm{e}^{i\delta_{\ell k}}\sin\delta_{\ell k}$$

Low-energy limit

- From behavior of phase shifts in this limit

$$\langle k|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle \to \frac{2\hbar^2}{m\pi}\,a\;\delta_{\ell,0}, \qquad \text{for } k \to 0$$

- also for

$$\langle k'|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle \to \frac{2\hbar^2}{m\pi} a\delta_{\ell,0}$$

- expansion for small momenta should yield the same constant
- In plane-wave basis

$$\langle \boldsymbol{k}'|\,\tilde{\mathcal{T}}\,|\boldsymbol{k}\rangle = \sum_{\ell m \ell' m'} \langle k'\ell' m'|\,\tilde{\mathcal{T}}\,|k\ell m\rangle\, Y_{\ell' m'}(\hat{k}')Y^*_{\ell m}(\hat{k})$$

$$= \sum_{\ell} \langle k'|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle\,\frac{2\ell+1}{4\pi} P_\ell(\omega)$$

- using rotational invariance

$$\langle k'\ell' m'|\,\tilde{\mathcal{T}}\,|k\ell m\rangle = \delta_{\ell,\ell'}\delta_{m,m'}\,\langle k'|\,\tilde{\mathcal{T}}^{\ell}\,|k\rangle$$

- So plane-wave half-on-shell matrix element isotropic

$$\langle \boldsymbol{k}'|\,\tilde{\mathcal{T}}\,|\boldsymbol{k}\rangle \to \frac{\hbar^2}{2\pi^2 m} a$$

Contact force

- Corresponding effective interaction in coordinate space from

$$(\boldsymbol{r}_1'\boldsymbol{r}_2'|\tilde{\mathcal{T}}|\boldsymbol{r}_1\boldsymbol{r}_2) = \int \frac{d\boldsymbol{k}_1}{(2\pi)^{3/2}}\frac{d\boldsymbol{k}_2}{(2\pi)^{3/2}}\frac{d\boldsymbol{k}_1'}{(2\pi)^{3/2}}\frac{d\boldsymbol{k}_2'}{(2\pi)^{3/2}} \mathrm{e}^{i(\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\boldsymbol{k}_2\cdot\boldsymbol{r}_2-\boldsymbol{k}_1'\cdot\boldsymbol{r}_1'-\boldsymbol{k}_2'\cdot\boldsymbol{r}_2')} \langle \boldsymbol{k}_1'\boldsymbol{k}_2'|\, \tilde{\mathcal{T}}\, |\boldsymbol{k}_1\boldsymbol{k}_2\rangle$$

- Introducing $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2,\ \boldsymbol{K}' = \boldsymbol{k}_1' + \boldsymbol{k}_2';\ \ \boldsymbol{k} = \dfrac{\boldsymbol{k}_1 - \boldsymbol{k}_2}{2},\ \boldsymbol{k}' = \dfrac{\boldsymbol{k}_1' - \boldsymbol{k}_2'}{2}$
- and employing momentum conservation $(\boldsymbol{k}_1'\boldsymbol{k}_2'|\tilde{\mathcal{T}}|\boldsymbol{k}_1\boldsymbol{k}_2) = \delta(\boldsymbol{K} - \boldsymbol{K}')\, \langle \boldsymbol{k}'|\, \tilde{\mathcal{T}}\, |\boldsymbol{k}\rangle$
- Substitution of low-energy limit then yields

$$(\boldsymbol{r}_1'\boldsymbol{r}_2'|\tilde{\mathcal{T}}|\boldsymbol{r}_1\boldsymbol{r}_2) = \frac{4\pi\hbar^2 a}{m}\delta(\boldsymbol{r}_1 - \boldsymbol{r}_1')\delta(\boldsymbol{r}_2 - \boldsymbol{r}_2')\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- Form of local zero-range interaction

$$\tilde{\mathcal{T}}(\boldsymbol{r}_1 - \boldsymbol{r}_2) = g\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

- with $g = \dfrac{4\pi\hbar^2 a}{m}$
- So previous assertions are justified

Gross-Pitaevskii (GP) equation

- Dilute system: average interparticle spacing $\rho^{-1/3}$ large compared to magnitude of scattering length, or $\rho|a|^3 << 1$
- Previous discussion suggests that HB mean-field can be applied in dilute case: replace V (even if strong) with the pseudo potential
- HB potential then becomes $W_{HB}(\boldsymbol{r}) = g(N-1)|\phi_c(\boldsymbol{r})|^2 \approx gN|\phi_c(\boldsymbol{r})|^2$
- HB equation $\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\boldsymbol{r})\right]\phi_c(\boldsymbol{r}) + gN|\phi_c(\boldsymbol{r})|^2\phi_c(\boldsymbol{r}) = \mu\phi_c(\boldsymbol{r})$
- Looks like nonlinear Schrödinger equation and is referred to as the time-independent Gross-Pitaevskii equation
- Condensate orbital also minimizes (with $\int d\boldsymbol{r}|\phi_c(\boldsymbol{r})|^2 = 1$)

$$E_0^{GP}/N = \int d\boldsymbol{r}\left(\frac{\hbar^2}{2m}|\nabla\phi_c(\boldsymbol{r})|^2 + U(\boldsymbol{r})|\phi_c(\boldsymbol{r})|^2 + \frac{gN}{2}|\phi_c(\boldsymbol{r})|^4\right)$$

- Time-dependent GP equation

$$\left[-\frac{1}{2m}\nabla^2 + U(\boldsymbol{r};t)\right]\phi_c(\boldsymbol{r};t) + gN|\phi_c(\boldsymbol{r},\boldsymbol{t})|^2\phi_c(\boldsymbol{r};t) = i\hbar\frac{\partial}{\partial t}\phi_c(\boldsymbol{r};t)$$

- GP and GP(t) all that is needed to explain most data BEC

Confined bosons in harmonic traps

- Confining potential well approximated by HO
$$U(\boldsymbol{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$
- usually with cylindrical symmetry (cigar or pancake)
- Include interaction at the GP level
- Effect can be large even when the system is dilute
- Estimate: assume condensate wave function approximately HO ground state $\phi_{000}(\boldsymbol{r})$ (see Ch. 5)
- Central density of condensate $\rho(0) = N|\phi_c(0)|^2 \approx N\left(\frac{m\omega_{HO}}{\pi\hbar}\right)^{3/2} = \frac{N}{\pi^{3/2}a_{HO}^3}$
- Typical values: $10^3 < N < 10^6$, $|a| \sim 10^{-9}$m, $a_{HO} \sim 10^{-6}$m so $10^{-6} < \rho|a|^3 < 10^{-3}$
- So very dilute
- Consider mean-field potential $W_{HB}(0) = gN|\phi_c(0)|^2 \approx N\frac{a}{a_{HO}^3}\frac{4\hbar^2}{m\sqrt{\pi}}$

BEC in traps at the GP level

- Compare with HO energy scale $\hbar\omega_{HO} = \frac{\hbar^2}{m a_{HO}^2}$
- Ratio proportional to $u = N \frac{a}{a_{HO}}$
- Measure of strength of interaction effects
- For quoted values $1 < |u| < 10^3$
- So expect large deviations of GP w.r.t. noninteracting profile
- Example u~125

  $a = 2.75$ nm and a_{HO}=1.76 μm
- Column density $\bar{\rho}(z) = \int dy \rho(0, y, z)$
- 8×10^4 ^{23}Na atoms
- Trap $\omega_x = \omega_y = 2050$ rad/s and $\omega_z = 170$ rad/s
- Reduction of 12 w.r.t. HO only