IPM for fermions in finite systems

- IPM = independent particle model
- Only consider Pauli principle
- Localized fermions (for now)
- Examples
- Hamiltonian many-body problem: $\hat{H} = \hat{T} + \hat{V} = \hat{H}_0 + \hat{H}_1$
- with $\hat{H}_0 = \hat{T} + \hat{U}$
- and $\hat{H}_1 = \hat{V} - \hat{U}$
- Suitably chosen auxiliary one-body potential U
- Many-body problem can be solved for $\hat{H}_0$!!
- Also works with fixed external potential U_{ext}

$$\hat{H} = \hat{T} + \hat{U}_{ext} + \hat{V} = \hat{H}_0 + \hat{H}_1$$

Role of U

- Can be chosen to minimize effect of two-body interaction
- Ground state of total Hamiltonian may break a symmetry
  - Spontaneous magnetization
- Can speed up convergence of perturbation expansion in $\hat{H}_1$

- Spherical symmetry: sp problem straightforward but may have to be done numerically
- Assume solved: e.g. 3D-harmonic oscillator in nuclear physics

$$H_0 \left|\lambda\right\rangle = (T + U) \left|\lambda\right\rangle = \varepsilon_\lambda \left|\lambda\right\rangle$$

- For nuclei $\left|\lambda\right\rangle = \left|n(\ell\tfrac{1}{2})jm_j\right\rangle$
- For atoms (include Coulomb attraction to nucleus)

$$\left|\lambda\right\rangle = \left|n\ell m_\ell \tfrac{1}{2} m_s\right\rangle$$

Use second quantization

- Consider in the $\{|\lambda\rangle\}$ basis (discrete sums for simplicity)

$$\begin{aligned}\hat{H}_0 &= \sum_{\lambda\lambda'} \langle\lambda| (T+U) |\lambda'\rangle a_\lambda^\dagger a_{\lambda'} \\ &= \sum_{\lambda\lambda'} \varepsilon_{\lambda'} \, \delta_{\lambda,\lambda'} \, a_\lambda^\dagger a_{\lambda'} = \sum_\lambda \varepsilon_\lambda a_\lambda^\dagger a_\lambda\end{aligned}$$

- All many-body eigenstates of $\hat{H}_0$ are of the form

$$|\Phi_n^N\rangle = |\lambda_1\lambda_2...\lambda_N\rangle = a_{\lambda_1}^\dagger a_{\lambda_2}^\dagger ... a_{\lambda_N}^\dagger |0\rangle$$

- with eigenvalue

$$E_n^N = \sum_{i=1}^N \varepsilon_{\lambda_i}$$

Explicitly

- Employ $$\left[\hat{H}_0, a^\dagger_{\lambda_i}\right] = \varepsilon_{\lambda_i} a^\dagger_{\lambda_i}$$

- and therefore

$$\begin{aligned}
\hat{H}_0 \left|\lambda_1\lambda_2\lambda_3...\lambda_N\right\rangle &= \hat{H}_0 a^\dagger_{\lambda_1} a^\dagger_{\lambda_2}...a^\dagger_{\lambda_N} \left|0\right\rangle \\
&= [\hat{H}_0, a^\dagger_{\lambda_1}] a^\dagger_{\lambda_2}...a^\dagger_{\lambda_N} \left|0\right\rangle + a^\dagger_{\lambda_1} \hat{H}_0 a^\dagger_{\lambda_2}...a^\dagger_{\lambda_N} \left|0\right\rangle \\
&= [\hat{H}_0, a^\dagger_{\lambda_1}] a^\dagger_{\lambda_2}...a^\dagger_{\lambda_N} \left|0\right\rangle + a^\dagger_{\lambda_1} [\hat{H}_0, a^\dagger_{\lambda_2}]...a^\dagger_{\lambda_N} \left|0\right\rangle + ... + a^\dagger_{\lambda_1} a^\dagger_{\lambda_2}...[\hat{H}_0, a^\dagger_{\lambda_N}] \left|0\right\rangle \\
&= \left\{\sum_{i=1}^{N} \varepsilon_{\lambda_i}\right\} \left|\lambda_1\lambda_2\lambda_3...\lambda_N\right\rangle
\end{aligned}$$

- Corresponding many-body problem solved!
- Ground state $\left|\Phi_0^N\right\rangle = \prod_{\lambda_i \leq F} a^\dagger_{\lambda_i} \left|0\right\rangle$
- Fermi sea $\Rightarrow$ F

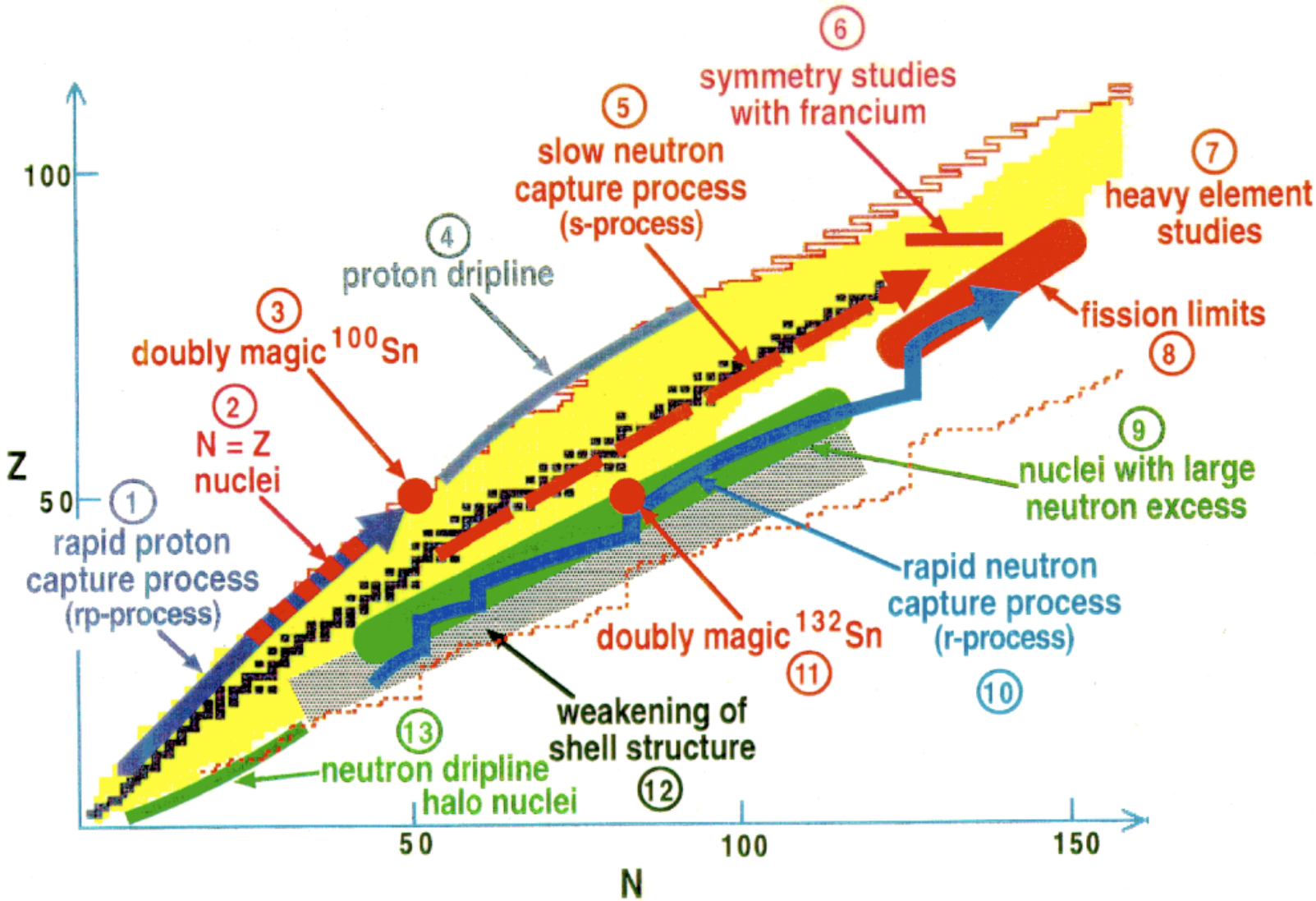
Nucleons in nuclei

- Atoms: shell closures at 2,10,18,36,54,86
- Similar features observed in nuclei
- Notation:
  - # of neutrons N
  - # of protons Z
  - # of nucleons $A = N + Z$
- Equivalent of ionization energy: separation energy
  - for protons $S_p(N,Z) = B(N,Z) - B(N,Z-1)$
  - for neutrons $S_n(N,Z) = B(N,Z) - B(N-1,Z)$
  - binding energy

$$M(N,Z) = \frac{E(N,Z)}{c^2} = N\ m_n\ +\ Z\ m_p\ -\ \frac{B(N,Z)}{c^2}$$

Chart of nuclides

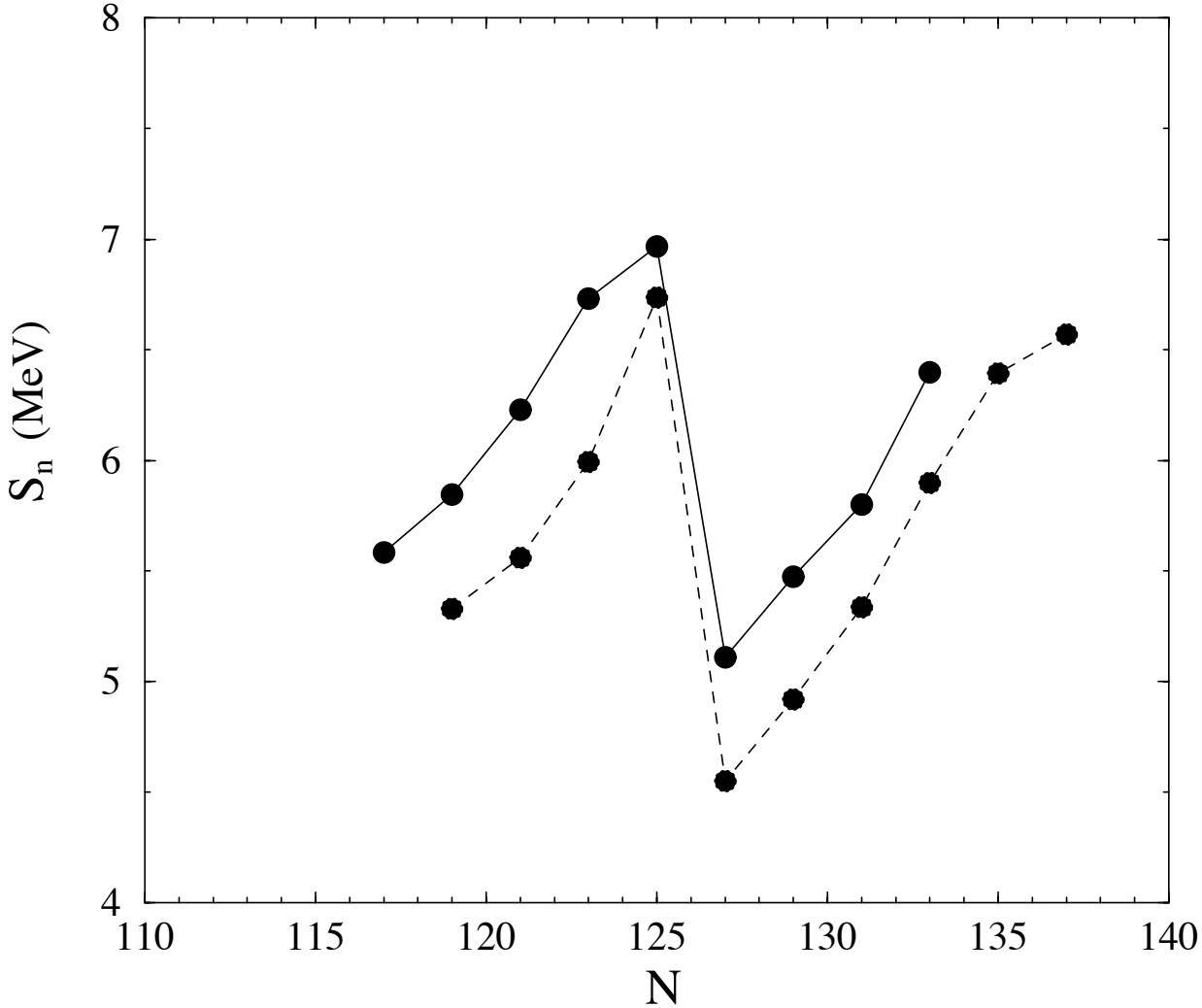
- Lots of nuclei and lots to be discovered



- Links to astrophysics

Shell closure at N=126

- Odd-even effect: plot only even Z



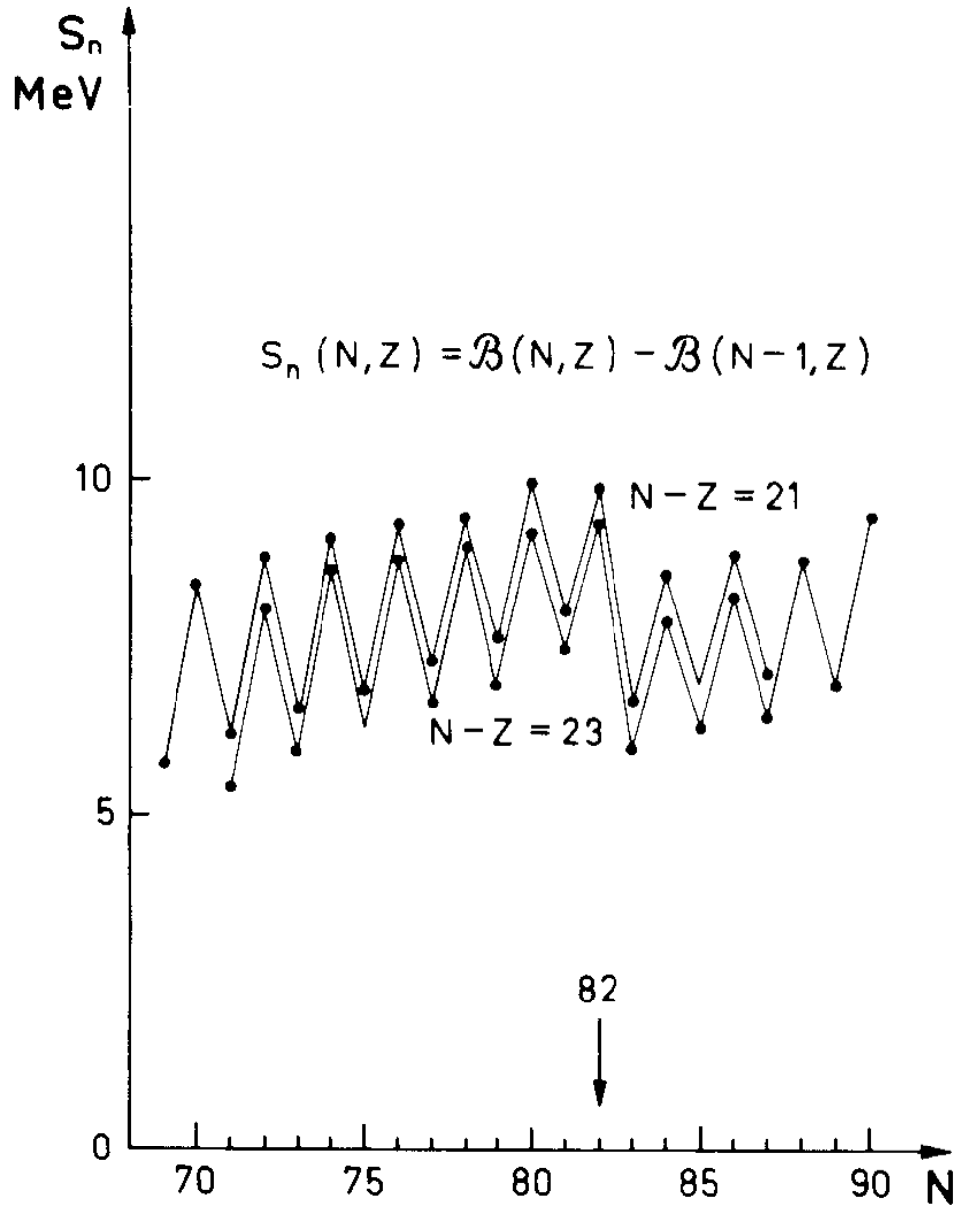
Solid: N-Z=41

Dashed: N-Z=43

- Also at other values N and Z

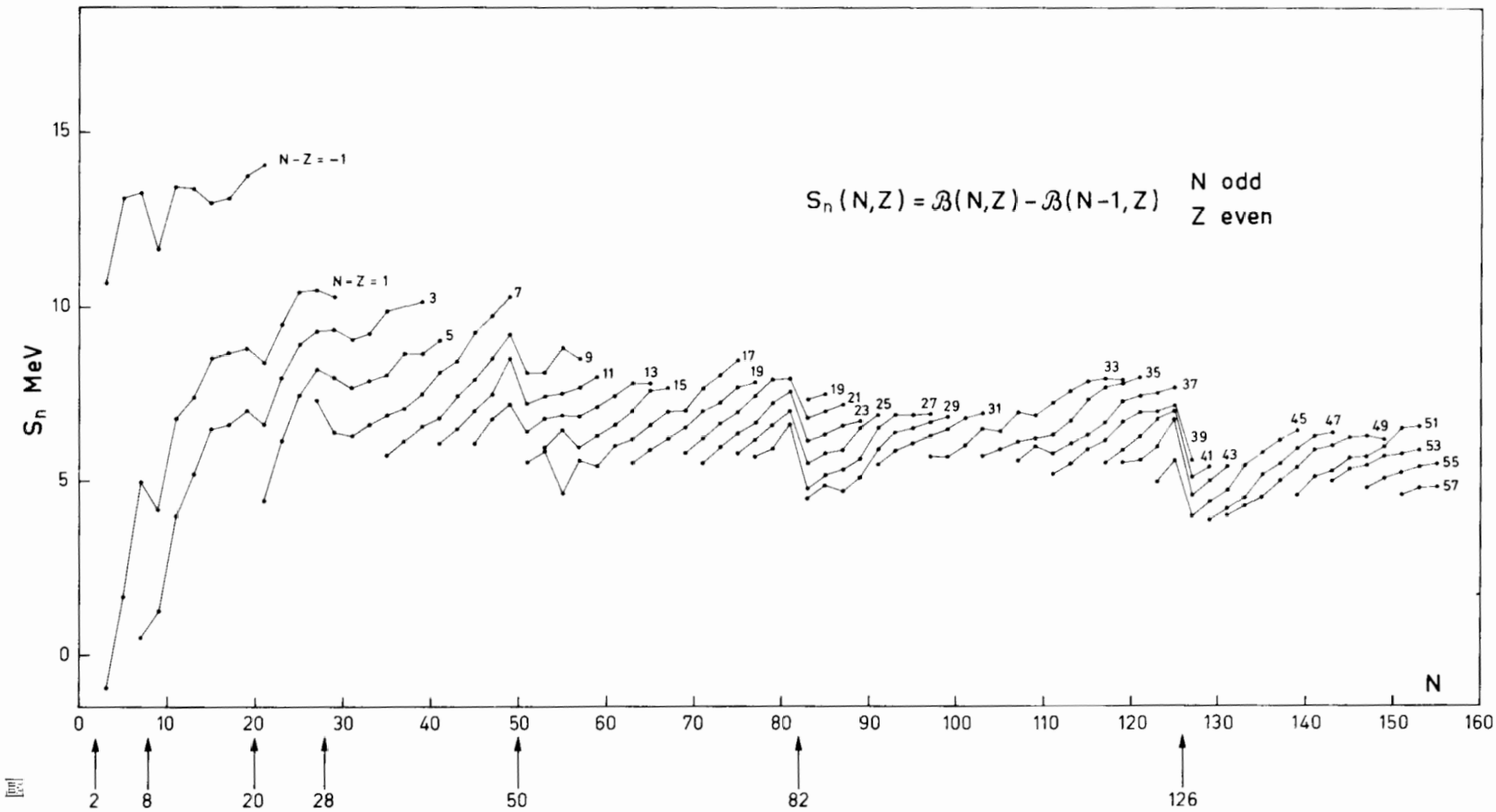
Illustration of odd-even effect

- from Bohr & Mottelson Vol.1 (BM1)



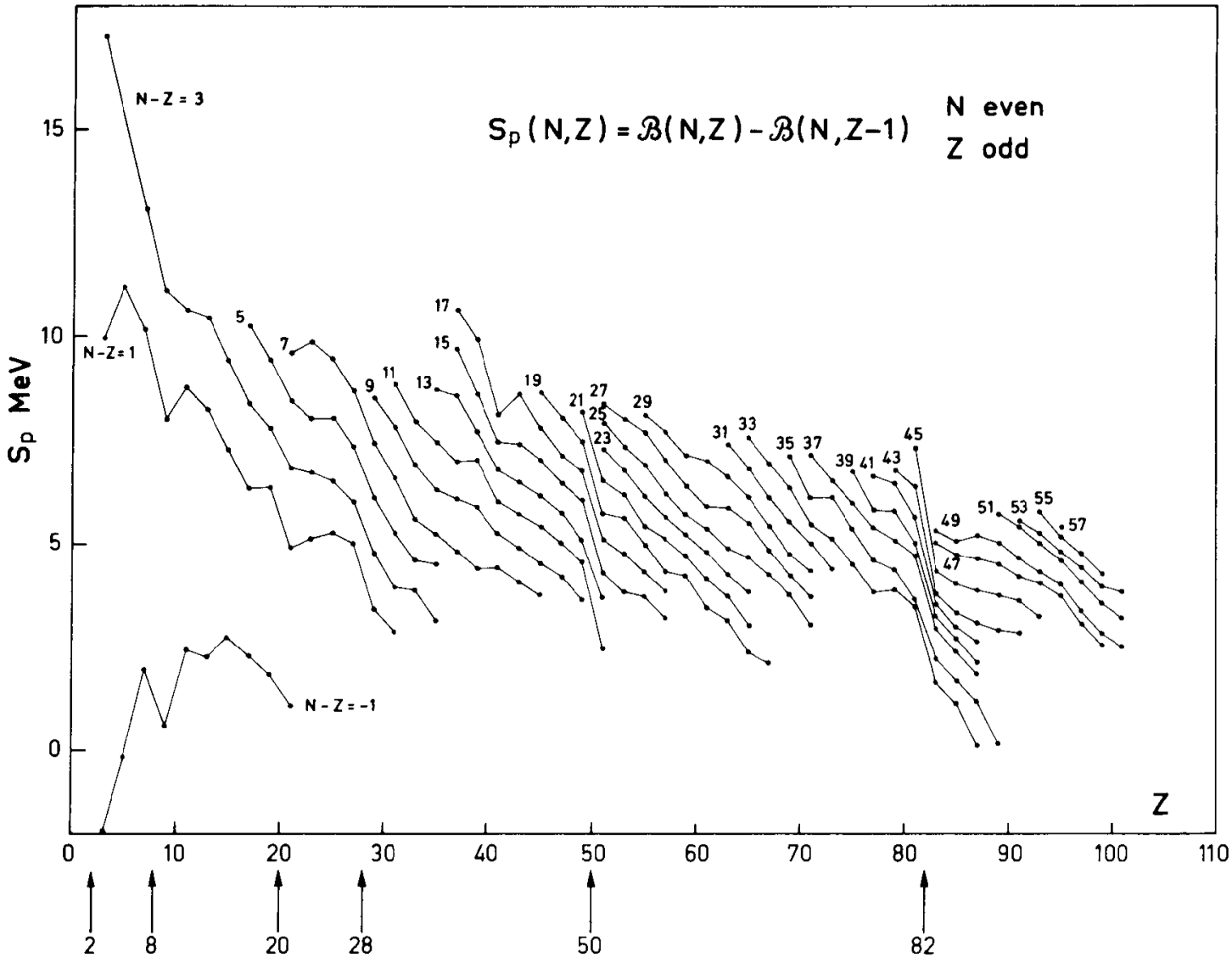
Neutrons

- BM1 figure



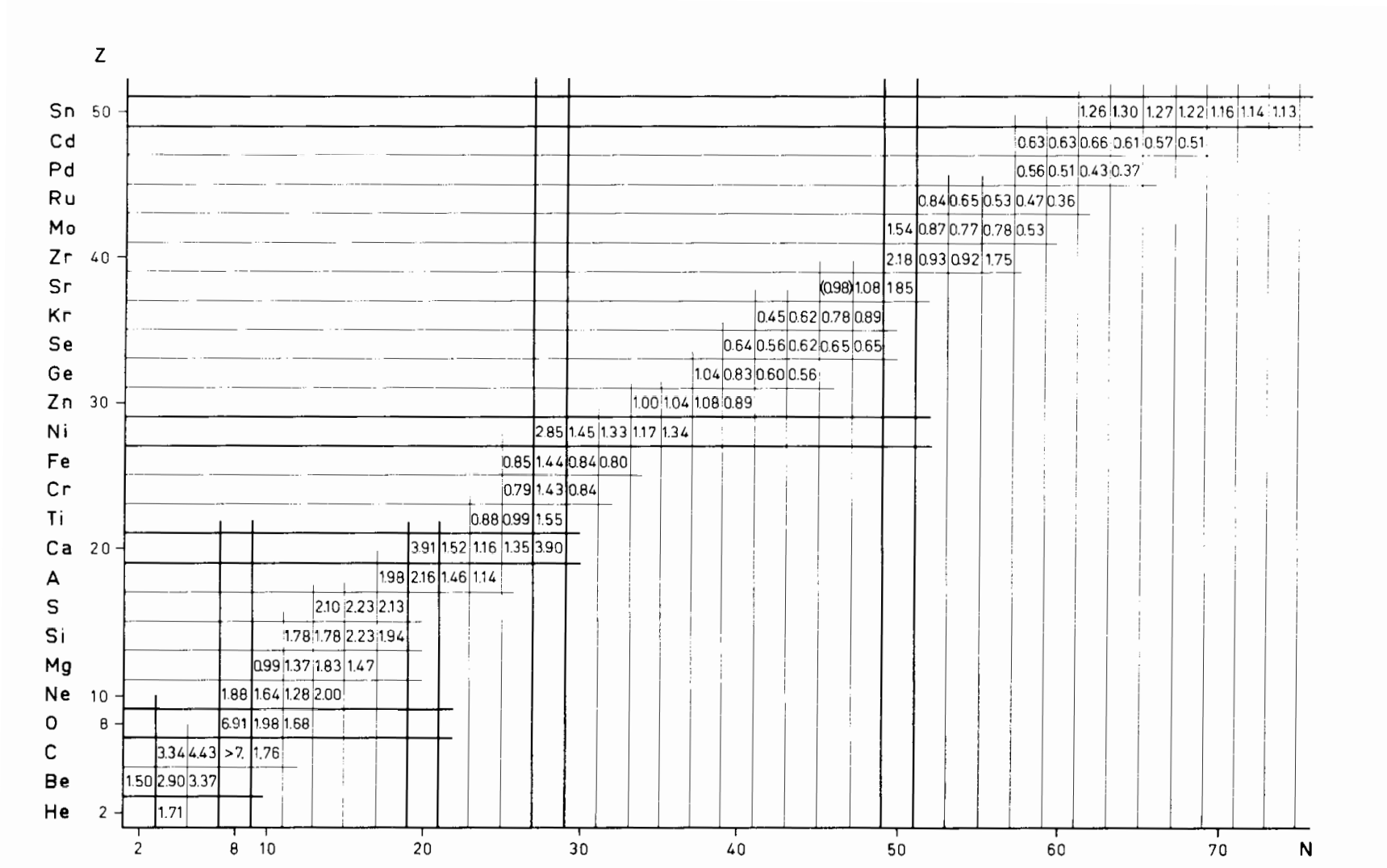
Protons

- BM1 figure



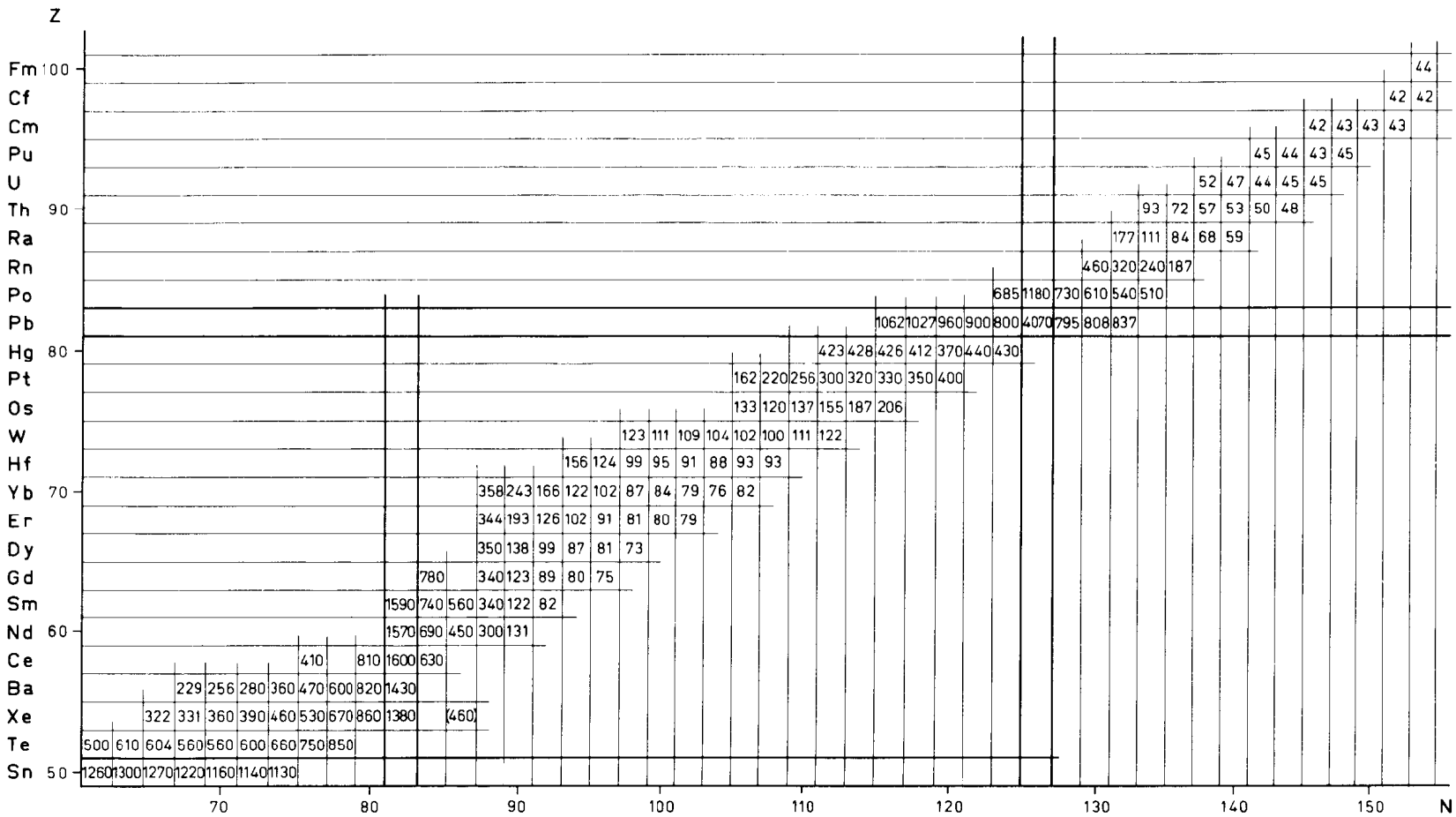
Systematics excitation energies in even-even nuclei

- Ground states 0^+
- First excited state almost always 2^+
- Excitation energy in MeV



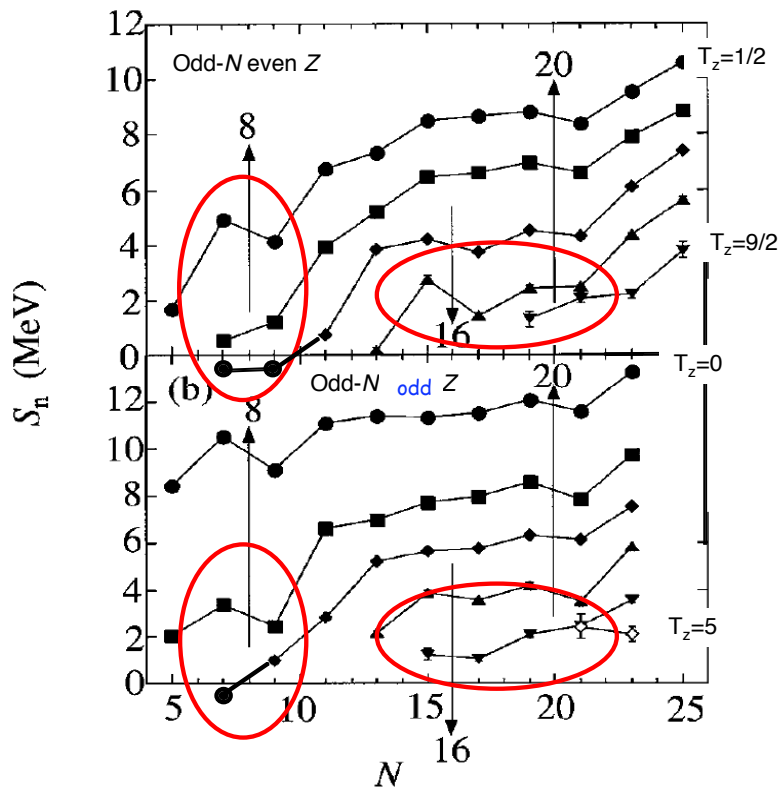
Heavy nuclei

- Magic numbers for nuclei near stability:
  - Z=2, 8, 20, 28, 50, 82
  - N=2, 8, 20, 28, 50, 82, 126



Nuclear shell structure

- Ground-state spins and parity of odd nuclei provide further evidence of "magic numbers"
- Character of magic numbers may change far from stability (hot)

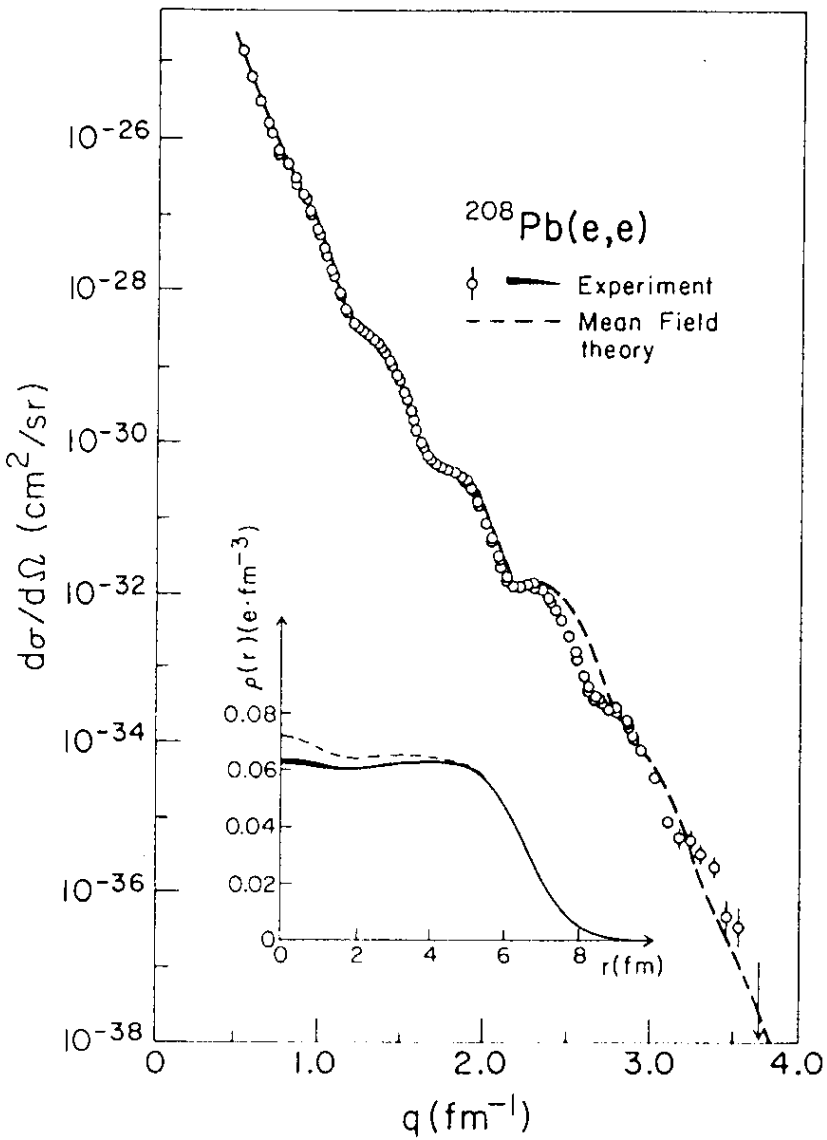


A. Ozawa *et al*., Phys. Rev. Lett. 84, 5493 (2000)

- N=20 may disappear and N=16 may appear

Empirical potential

- Analogy to atoms suggests finding a sp potential ⇒ shells + IPM
- Difference(s) with atoms?
- Properties of empirical potential
  - overall?
  - size?
  - shape?
- Consider nuclear charge density



Frois & Papanicolas, Ann. Rev. Nucl. Part. Sci. **37**, 133 (1987)

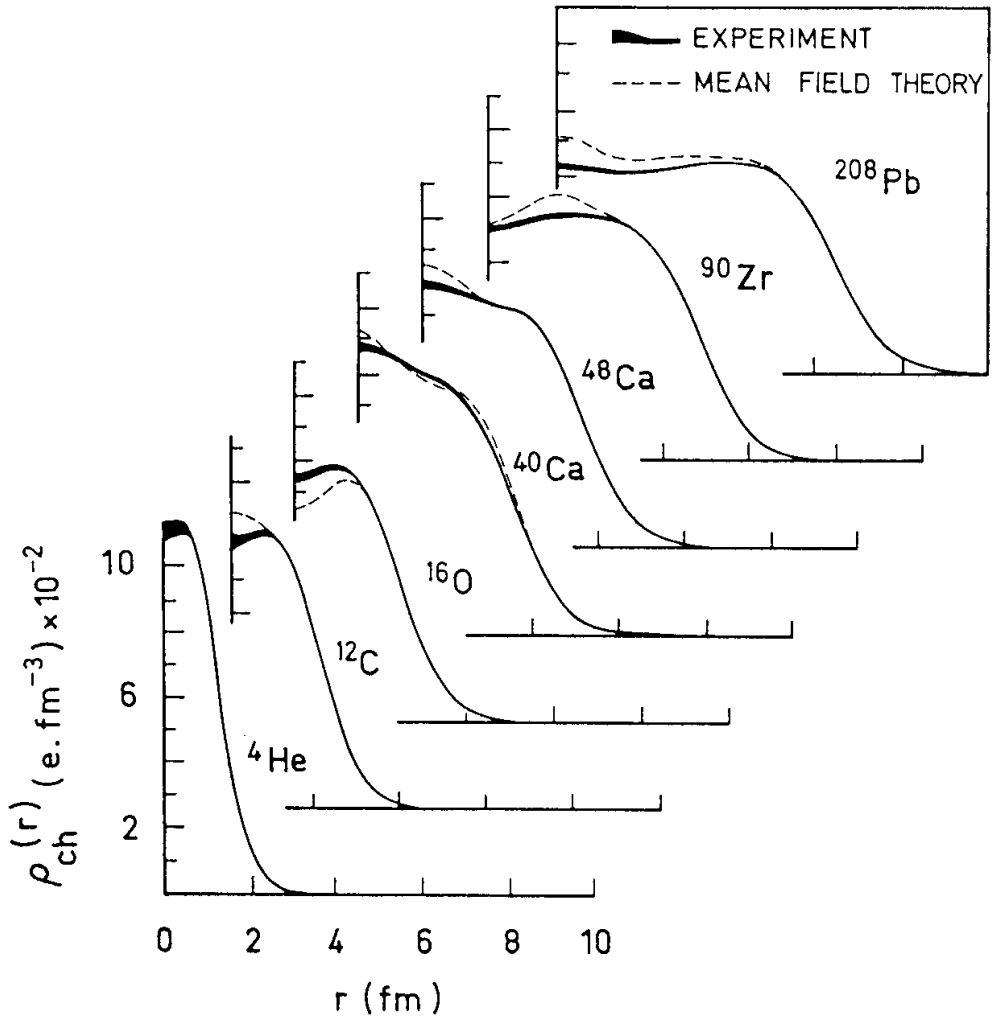
Nuclear density distribution

- Central density (A/Z* charge density) about the same for nuclei heavier than ^{16}O, corresponding to 0.16 nucleons/fm^3
- Important quantity
- Shape roughly represented by
- $$\rho_{ch}(r) = \frac{\rho_0}{1 + \exp\left(\frac{r-c}{z}\right)}$$

  $$c \approx 1.07 A^{\frac{1}{3}}\,\mathrm{fm}$$

  $$z \approx 0.55\mathrm{fm}$$

- Potential similar shape



Empirical potential

- Bohr Mottelson Vol.1

$$U = V f(r) + V_{\ell s} \left(\frac{\boldsymbol{\ell} \cdot \boldsymbol{s}}{\hbar^2} \right) r_0^2 \frac{1}{r} \frac{d}{dr} f(r)$$

- Central part roughly follows shape of density

$$f(r) = \left[1 + \exp\left(\frac{r - R}{a} \right) \right]^{-1}$$

- Woods-Saxon form
- Depth $V = \left[-51 \pm 33 \left(\frac{N - Z}{A} \right) \right] \text{MeV}$

  + neutrons

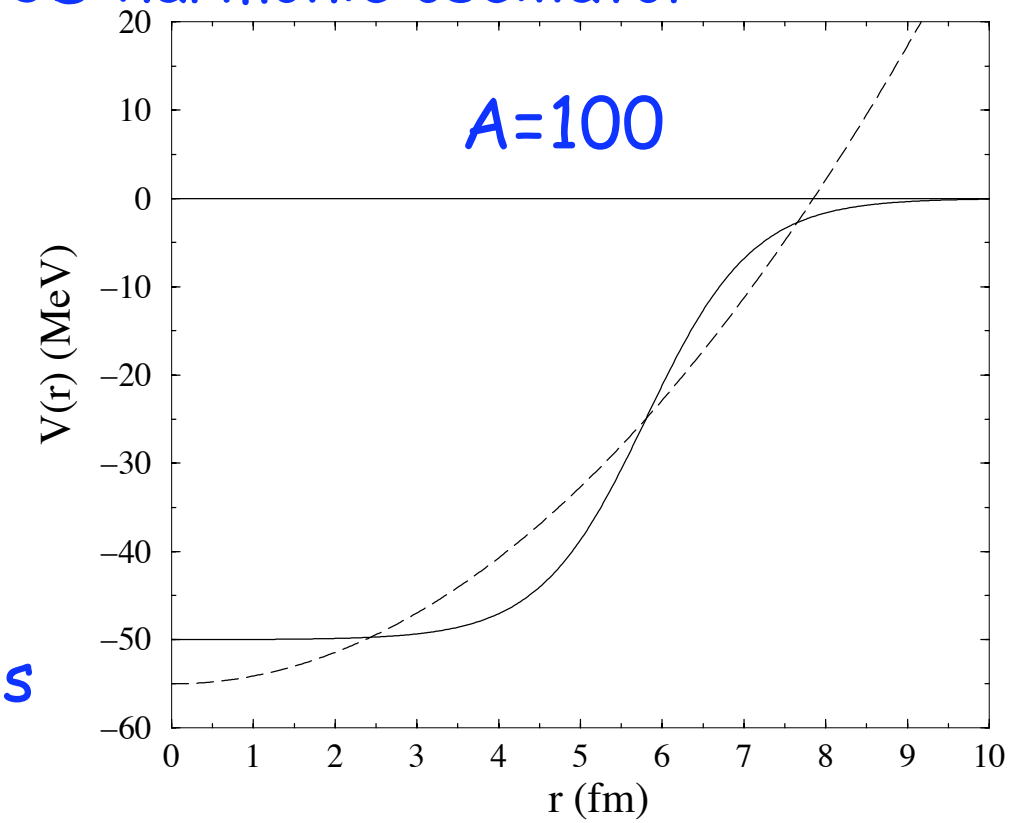
  - protons
- radius $R = r_0 \, A^{1/3}$ with $r_0 = 1.27 \text{ fm}$
- diffuseness $a = 0.67 \text{ fm}$

Analytically solvable alternative

- Woods-Saxon (WS) generates finite number of bound states
- IPM: fill lowest levels ⇒ nuclear shells ⇒ magic numbers
- reasonably approximated by 3D harmonic oscillator

$$U_{HO}(r) = \frac{1}{2} m\omega^2 r^2 - V_0$$

$$H_0 = \frac{\boldsymbol{p}^2}{2m} + U_{HO}(r)$$



- Eigenstates in spherical basis

$$H_{HO}\,|n\ell m_\ell m_s\rangle = \left(\hbar\omega(2n+\ell+\tfrac{3}{2}) - V_0\right)|n\ell m_\ell m_s\rangle$$

Harmonic oscillator

- Filling of oscillator shells
- # of quanta $N = 2n + \ell$

N	n	ℓ	# of particles	"magic #"	parity
0	0	0	2	2	+
1	0	1	6	8	-
2	1	0	2		+
2	0	2	10	20	+
3	1	1	6		-
3	0	3	14	40	-
4	2	0	2		+
4	1	2	10		+
4	0	4	18	70	+

Need for another type of sp potential

- 1949 Mayer and Jensen suggest the need of a spin-orbit term
- Requires a coupled basis

$$|n(\ell s)jm_j\rangle = \sum_{m_\ell m_s} |n\ell m_\ell m_s\rangle \, (\ell \; m_\ell \; s \; m_s | \; j \; m_j)$$

- Use $\boldsymbol{\ell}\cdot\boldsymbol{s} = \frac{1}{2}(\boldsymbol{j}^2 - \boldsymbol{\ell}^2 - \boldsymbol{s}^2)$ to show that these are eigenstates

$$\frac{\boldsymbol{\ell}\cdot\boldsymbol{s}}{\hbar^2}|n(\ell s)jm_j\rangle = \tfrac{1}{2}\left(j(j+1) - \ell(\ell+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right)|n(\ell s)jm_j\rangle$$

- For $j = \ell + \frac{1}{2}$ eigenvalue $\frac{1}{2}\ell$
- while for $j = \ell - \frac{1}{2}$ $-\frac{1}{2}(\ell+1)$
- so SO splits these levels! and more so with larger ℓ

Inclusion of SO potential and magic numbers

- Sign of SO?

$$V_{\ell s}\left(\frac{\boldsymbol{\ell}\cdot\boldsymbol{s}}{\hbar^2}\right) r_0^2 \frac{1}{r}\frac{d}{dr} f(r)$$

$$V_{\ell s} = -0.44V$$

- Consequence for

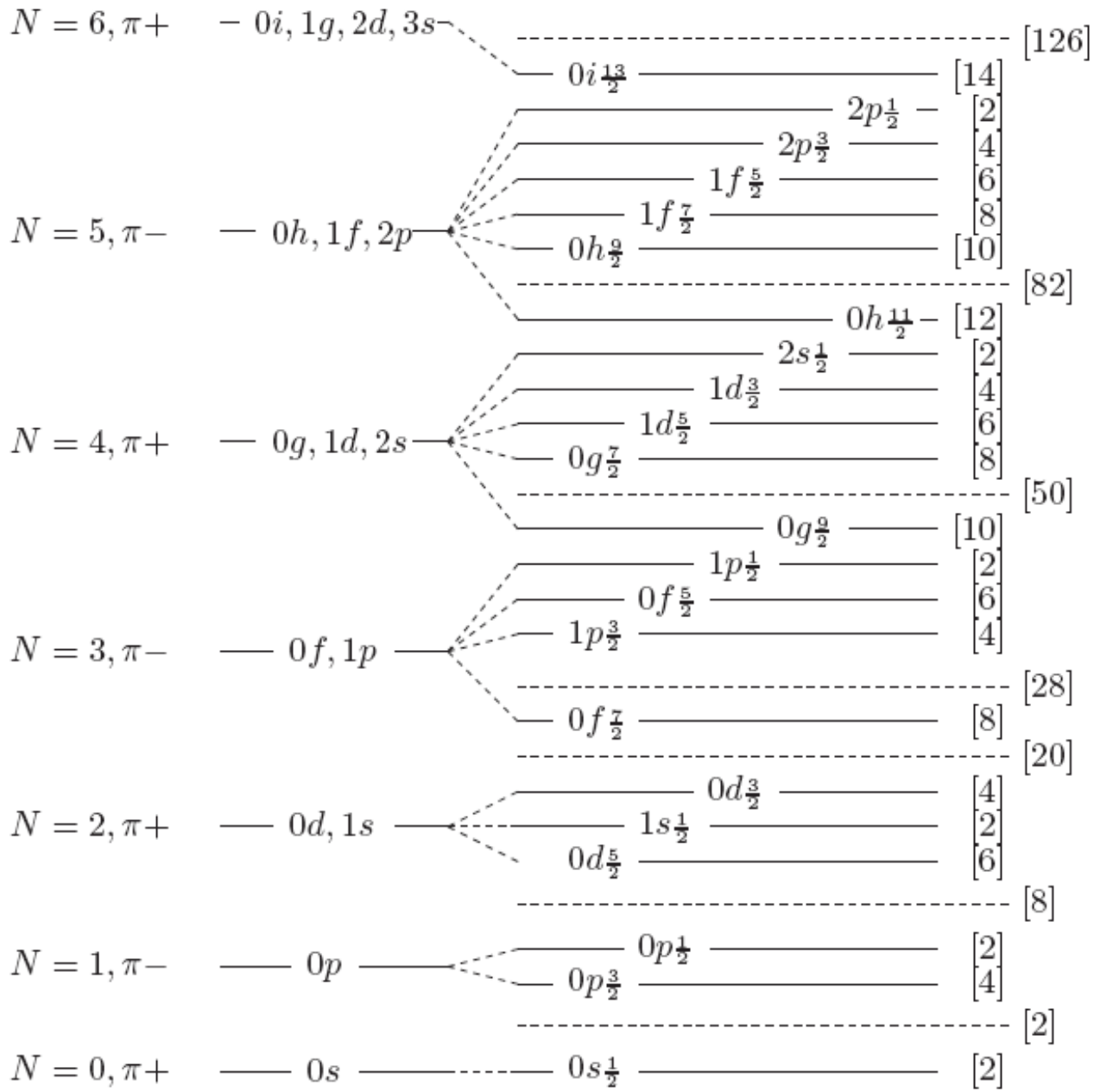
$$0f\,\tfrac{7}{2}$$

$$0g\,\tfrac{9}{2}$$

$$0h\,\tfrac{11}{2}$$

$$0i\,\tfrac{13}{2}$$

- Noticeably shifted
- Correct magic numbers!



^{208}Pb for example

- Empirical potential & sp energies

$$\hat{H}_0 \; a^\dagger_\alpha \, |^{208}\mathrm{Pb}_{g.s.}\rangle = \left[\varepsilon_\alpha + E(^{208}\mathrm{Pb}_{g.s.})\right] \; a^\dagger_\alpha \, |^{208}\mathrm{Pb}_{g.s.}\rangle$$

- A+1: "sp energies" $E_n^{A+1} - E_0^A$ directly from experiment
- A-1:

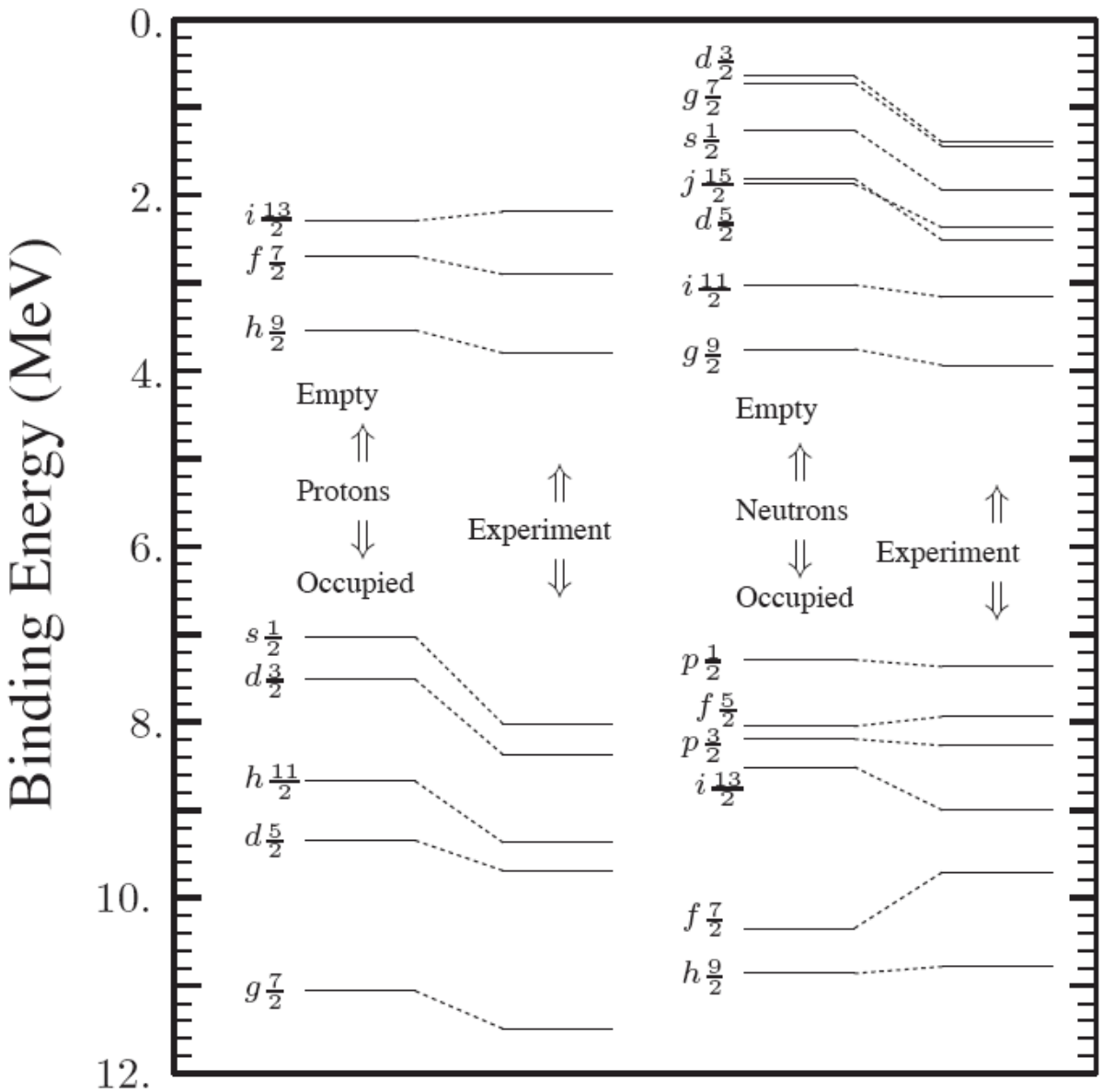
$$\hat{H}_0 \; a_\alpha \, |^{208}\mathrm{Pb}_{g.s.}\rangle = \left[E(^{208}\mathrm{Pb}_{g.s.}) - \varepsilon_\alpha\right] \; a_\alpha \, |^{208}\mathrm{Pb}_{g.s.}\rangle$$

- also directly from $E_0^A - E_n^{A-1}$

- Shell filling for nuclei near stability follows empirical potential

Comparison with experiment

- Now how to explain this potential …



Nucleon-nucleon interaction

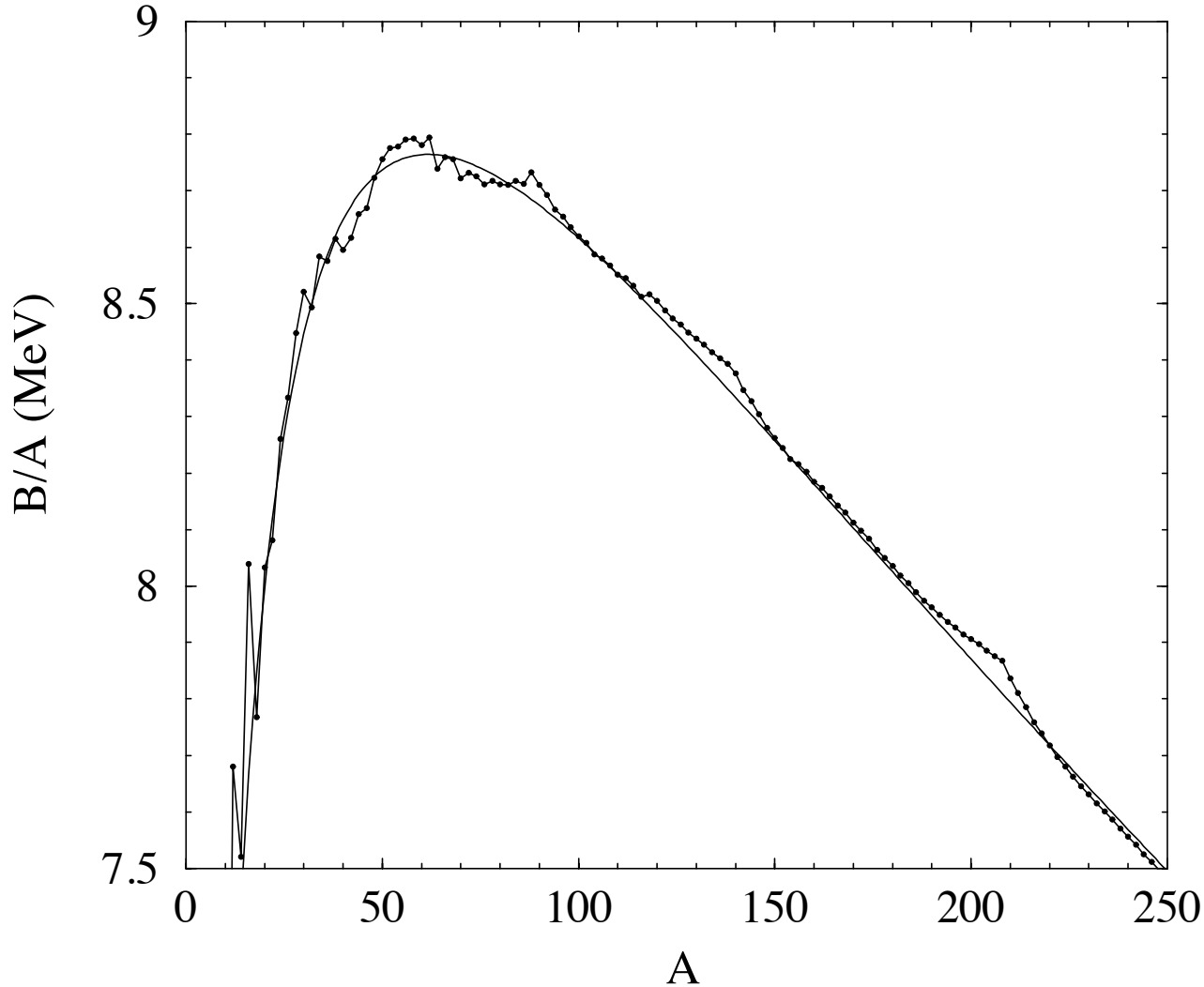
- Shell structure in nuclei and lots more to be explained on the basis of how nucleons interact with each other in free space

- QCD
- Lattice calculations
- Effective field theory
- Exchange of lowest bosonic states
- Phenomenology

- Realistic NN interactions: describe NN scattering data up to pion production threshold plus deuteron properties
- Note: extra energy scale from confinement of nucleons

Nuclear Matter

- Nuclear masses near stability $M(N,Z) = \frac{E(N,Z)}{c^2} = N\ m_n\ +\ Z\ m_p\ -\ \frac{B(N,Z)}{c^2}$
- Data
- Each A most stable N,Z pair
- Where fission?
- Where fusion?



Nuclear Matter

- Smooth curve

$$B = b_{vol}A - b_{surf}A^{2/3} - \frac{1}{2}b_{sym}\frac{(N-Z)^2}{A} - \frac{3}{5}\frac{Z^2e^2}{R_c}$$

- volume $b_{vol} = 15.56 \text{ MeV}$
- surface $b_{surf} = 17.23 \text{ MeV}$
- symmetry $b_{sym} = 46.57 \text{ MeV}$
- Coulomb $R_c = 1.24A^{1/3} \text{ fm}$

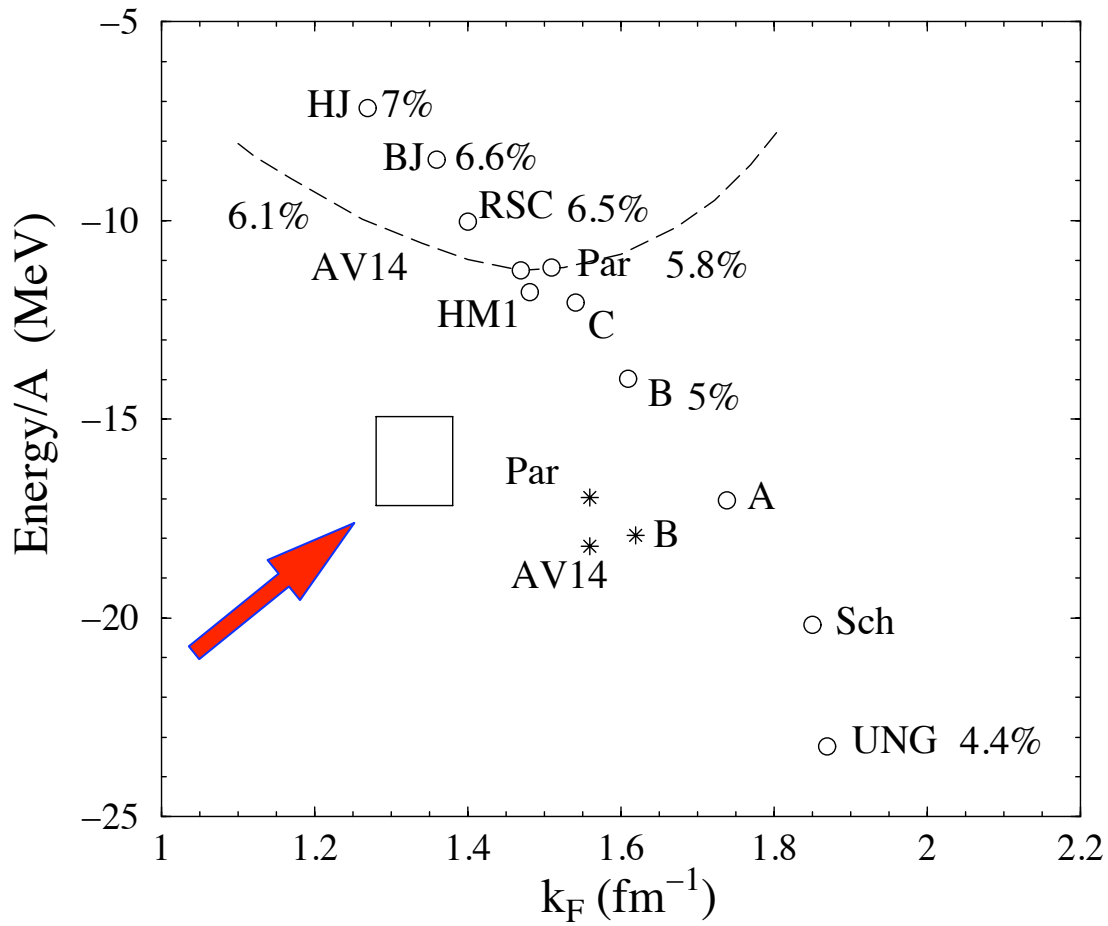
Great interest in limit: N=Z; no Coulomb; A ⇒∞

Two most important numbers in Nuclear Physics

$$\frac{B}{A} \approx 16 \text{ MeV} \qquad \rho_0 \approx 0.16 \text{ fm}^3$$

Saturation problem of nuclear matter

Given V_{NN} ⇒ explain correct minimum of E/A in nuclear matter as a function of density inside empirical box



Describe the infinite system of neutrons

⇒ properties of neutron stars

Isospin

- Shell closures for N and Z the same!!
- Also $m_n c^2 \approx m_p c^2$ 939.56 MeV vs. 938.27 MeV
- So strong interaction Hamiltonian (QCD) invariant for p ⇔ n
- But weak and electromagnetic interactions are not
- Strong interaction dominates ⇒ consequences
- Notation (for now) $p_\alpha^\dagger$ adds proton
  $n_\alpha^\dagger$ adds neutron
- Anticommutation relations $\{p_\alpha^\dagger, p_\beta\} = \delta_{\alpha,\beta}$
  $\{n_\alpha^\dagger, n_\beta\} = \delta_{\alpha,\beta}$

Isospin

- Z proton & N neutron state

$$|\alpha_1\alpha_2...\alpha_Z;\beta_1\beta_2...\beta_N\rangle = p^\dagger_{\alpha_1}p^\dagger_{\alpha_2}...p^\dagger_{\alpha_Z}n^\dagger_{\beta_1}n^\dagger_{\beta_2}...n^\dagger_{\beta_N}\,|0\rangle$$

- Exchange all p with n $\quad \hat{T}^+ = \sum_\alpha p^\dagger_\alpha n_\alpha$
- and vice versa $\quad \hat{T}^- = \sum_\alpha n^\dagger_\alpha p_\alpha$
- Expect $\quad [\hat{H}_S, \hat{T}^\pm] = 0$
- Consider

$$\begin{aligned}\hat{T}_3 &= \frac{1}{2}[\hat{T}^+,\hat{T}^-] = \frac{1}{2}\sum_{\alpha\beta}\left(p^\dagger_\alpha n_\alpha n^\dagger_\beta p_\beta - n^\dagger_\beta p_\beta p^\dagger_\alpha n_\alpha\right)\\ &= \frac{1}{2}\sum_{\alpha\beta}\left(p^\dagger_\alpha p_\beta\delta_{\alpha,\beta} - n^\dagger_\beta n_\alpha\delta_{\alpha,\beta}\right) = \frac{1}{2}\sum_\alpha\left(p^\dagger_\alpha p_\alpha - n^\dagger_\alpha n_\alpha\right)\end{aligned}$$

- will also commute with H_S

Isospin

- Check $[\hat{T}_3, \hat{T}^{\pm}] = \pm\hat{T}^{\pm}$
- Then operators

$$\hat{T}_1 = \tfrac{1}{2}\left(\hat{T}^+ + \hat{T}^-\right)$$
$$\hat{T}_2 = \tfrac{1}{2i}\left(\hat{T}^+ - \hat{T}^-\right)$$
$$\hat{T}_3$$

obey the same algebra as J_x, J_y, J_z

so spectrum identical and $\hat{H}_S, \hat{\boldsymbol{T}}^2, \hat{T}_3$ simultaneously diagonal !

proton $|\boldsymbol{r} m_s\rangle_p = |\boldsymbol{r} m_s m_t = \tfrac{1}{2}\rangle$

neutron $|\boldsymbol{r} m_s\rangle_n = |\boldsymbol{r} m_s m_t = -\tfrac{1}{2}\rangle$

For this doublet $\boldsymbol{T}^2 |\boldsymbol{r} m_s m_t\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)|\boldsymbol{r} m_s m_t\rangle$

and $T_3 |\boldsymbol{r} m_s m_t\rangle = m_t |\boldsymbol{r} m_s m_t\rangle$

States with total isospin constructed as for angular momentum