IPM for fermions in finite systems

• IPM = independent particle model
• Only consider Pauli principle
• Localized fermions (for now)
• Examples
• Hamiltonian many-body problem: \( \hat{H} = \hat{T} + \hat{V} = \hat{H}_0 + \hat{H}_1 \)
  
  \begin{align*}
  \hat{H}_0 &= \hat{T} + \hat{U} \\
  \hat{H}_1 &= \hat{V} - \hat{U}
  \end{align*}

• Suitably chosen auxiliary one-body potential \( U \)
• Many-body problem can be solved for \( \hat{H}_0 \) !!
• Also works with fixed external potential \( U_{ext} \)

\( \hat{H} = \hat{T} + \hat{U}_{ext} + \hat{V} = \hat{H}_0 + \hat{H}_1 \)
Role of $U$

- Can be chosen to minimize effect of two-body interaction
- Ground state of total Hamiltonian may break a symmetry
  - Spontaneous magnetization
- Can speed up convergence of perturbation expansion in $\hat{H}_1$

- Spherical symmetry: sp problem straightforward but may have to be done numerically
- Assume solved: e.g. 3D-harmonic oscillator in nuclear physics
  \[ H_0 |\lambda\rangle = (T + U) |\lambda\rangle = \varepsilon_\lambda |\lambda\rangle \]
- For nuclei \( |\lambda\rangle = |n(\ell \frac{1}{2})jm_j\rangle \)
- For atoms (include Coulomb attraction to nucleus)
  \( |\lambda\rangle = |n\ell m_\ell \frac{1}{2}m_s\rangle \)
Use second quantization

• Consider in the \( \{ |\lambda\rangle \} \) basis (discrete sums for simplicity)

\[
\hat{H}_0 = \sum_{\lambda\lambda'} \langle \lambda | (T + U) | \lambda' \rangle a_{\lambda}^{\dagger} a_{\lambda'}
\]

\[
= \sum_{\lambda\lambda'} \varepsilon_{\lambda'} \delta_{\lambda,\lambda'} a_{\lambda}^{\dagger} a_{\lambda'} = \sum_{\lambda} \varepsilon_{\lambda} a_{\lambda}^{\dagger} a_{\lambda}
\]

• All many-body eigenstates of \( \hat{H}_0 \) are of the form

\[
|\Phi_N \rangle = |\lambda_1 \lambda_2 ... \lambda_N \rangle = a_{\lambda_1}^{\dagger} a_{\lambda_2}^{\dagger} ... a_{\lambda_N}^{\dagger} |0\rangle
\]

• with eigenvalue

\[
E_N^n = \sum_{i=1}^{N} \varepsilon_{\lambda_i}
\]
Explicitly

- Employ
  \[
  \left[ \hat{H}_0, a_{\lambda_i}^\dagger \right] = \varepsilon_{\lambda_i} a_{\lambda_i}^\dagger
  \]

- and therefore
  \[
  \begin{align*}
  \hat{H}_0 |\lambda_1 \lambda_2 \lambda_3 ... \lambda_N\rangle &= \hat{H}_0 a_{\lambda_1}^\dagger a_{\lambda_2}^\dagger ... a_{\lambda_N}^\dagger |0\rangle \\
  &= \left[ \hat{H}_0, a_{\lambda_1}^\dagger \right] a_{\lambda_2}^\dagger ... a_{\lambda_N}^\dagger |0\rangle + a_{\lambda_1}^\dagger \hat{H}_0 a_{\lambda_2}^\dagger ... a_{\lambda_N}^\dagger |0\rangle \\
  &= \left[ \hat{H}_0, a_{\lambda_1}^\dagger \right] a_{\lambda_2}^\dagger ... a_{\lambda_N}^\dagger |0\rangle + a_{\lambda_1}^\dagger \left[ \hat{H}_0, a_{\lambda_2}^\dagger \right] ... a_{\lambda_N}^\dagger |0\rangle + ... + a_{\lambda_1}^\dagger a_{\lambda_2}^\dagger ... \left[ \hat{H}_0, a_{\lambda_N}^\dagger \right] |0\rangle \\
  &= \left\{ \sum_{i=1}^{N} \varepsilon_{\lambda_i} \right\} |\lambda_1 \lambda_2 \lambda_3 ... \lambda_N\rangle
  \end{align*}
  \]

- Corresponding many-body problem solved!

- Ground state
  \[
  |\Phi_0^N\rangle = \prod_{\lambda_i \leq F} a_{\lambda_i}^\dagger |0\rangle
  \]

- Fermi sea $\Rightarrow F$
Nucleons in nuclei

- Atoms: shell closures at 2, 10, 18, 36, 54, 86
- Similar features observed in nuclei
- Notation:
  - # of neutrons $N$
  - # of protons $Z$
  - # of nucleons $A = N + Z$

- Equivalent of ionization energy: separation energy
  - for protons $S_p(N, Z) = B(N, Z) - B(N, Z - 1)$
  - for neutrons $S_n(N, Z) = B(N, Z) - B(N - 1, Z)$
  - binding energy
    \[ M(N, Z) = \frac{E(N, Z)}{c^2} = N \ m_n + Z \ m_p - \frac{B(N, Z)}{c^2} \]
Chart of nuclides

- Lots of nuclei and lots to be discovered

- Links to astrophysics
Shell closure at $N=126$

- **Odd-even effect:** plot only even $Z$

- Also at other values $N$ and $Z$

[Graph showing shell closure with solid line for $N-Z=41$ and dashed line for $N-Z=43$.]
Illustration of odd-even effect

- from Bohr & Mottelson Vol.1 (BM1)

\[ S_n(N, Z) = B(N, Z) - B(N-1, Z) \]
Neutrons

- BM1 figure

\[ S_n(N, Z) = B(N, Z) - B(N-1, Z) \]

N odd
Z even

\[ S_n \text{ MeV} \]
Protons

- BM1 figure

\[ S_p(N,Z) = B(N,Z) - B(N,Z-1) \]

\text{N even, Z odd}

\text{N - Z = 3}

\text{N - Z = 1}

\text{N - Z = -1}
Systematics excitation energies in even-even nuclei

- Ground states $0^+$
- First excited state almost always $2^+$
- Excitation energy in MeV
Heavy nuclei

- Magic numbers for nuclei near stability:
  - $Z=2, 8, 20, 28, 50, 82$
  - $N=2, 8, 20, 28, 50, 82, 126$
Nuclear shell structure

- Ground-state spins and parity of odd nuclei provide further evidence of “magic numbers”
- Character of magic numbers may change far from stability (hot)


- N=20 may disappear and N=16 may appear
Empirical potential

- Analogy to atoms suggests finding a sp potential ⇒ shells + IPM

- Difference(s) with atoms?

- Properties of empirical potential
  - overall?
  - size?
  - shape?

- Consider nuclear charge density

Nuclear density distribution

- Central density ($A/Z^*$ charge density) about the same for nuclei heavier than $^{16}\text{O}$, corresponding to 0.16 nucleons/fm$^3$

- Important quantity

- Shape roughly represented by

\[ \rho_{ch}(r) = \frac{\rho_0}{1 + \exp \left( \frac{r-c}{z} \right)} \]

\[ c \approx 1.07A^{1/3}\text{fm} \]
\[ z \approx 0.55\text{fm} \]

- Potential similar shape
Empirical potential

• Bohr Mottelson Vol.1

\[ U = V f(r) + V_{ls} \left( \frac{\ell \cdot s}{\hbar^2} \right) r_0^2 \frac{1}{r} \frac{d}{dr} f(r) \]

• Central part roughly follows shape of density

\[ f(r) = \left[ 1 + \exp \left( \frac{r - R}{a} \right) \right]^{-1} \]

• Woods-Saxon form

• Depth \[ V = \left[ -51 \pm 33 \left( \frac{N - Z}{A} \right) \right] \text{ MeV} \]
  + neutrons
  - protons

• radius \[ R = r_0 A^{1/3} \text{ with } r_0 = 1.27 \text{ fm} \]

• diffuseness \[ a = 0.67 \text{ fm} \]
Analytically solvable alternative

- Woods-Saxon (WS) generates finite number of bound states
- IPM: fill lowest levels $\Rightarrow$ nuclear shells $\Rightarrow$ magic numbers
- reasonably approximated by 3D harmonic oscillator

$$U_{HO}(r) = \frac{1}{2} m \omega^2 r^2 - V_0$$

$$H_0 = \frac{\mathbf{p}^2}{2m} + U_{HO}(r)$$

- Eigenstates in spherical basis

$$H_{HO} |n \ell m_\ell m_s\rangle = (\hbar \omega (2n + \ell + \frac{3}{2}) - V_0) |n \ell m_\ell m_s\rangle$$
Harmonic oscillator

- Filling of oscillator shells
- \# of quanta \[ N = 2n + \ell \]

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<th>( n )</th>
<th>( \ell )</th>
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Need for another type of sp potential

• 1949 Mayer and Jensen suggest the need of a spin-orbit term

• Requires a coupled basis

\[ |n(\ell s) j m_j \rangle = \sum_{m_\ell m_s} |n l m_\ell m_s \rangle (\ell m_\ell s m_s | j m_j) \]

• Use \( \ell \cdot s = \frac{1}{2} (j^2 - \ell^2 - s^2) \) to show that these are eigenstates

\[ \frac{\ell \cdot s}{\hbar^2} |n(\ell s) j m_j \rangle = \frac{1}{2} (j(j+1) - \ell(\ell+1) - \frac{1}{2} (\frac{1}{2} + 1)) |n(\ell s) j m_j \rangle \]

• For \( j = \ell + \frac{1}{2} \) eigenvalue \( \frac{1}{2} \ell \)

• while for \( j = \ell - \frac{1}{2} \) \(-\frac{1}{2}(\ell + 1)\)

• so SO splits these levels! and more so with larger \( \ell \)
Inclusion of SO potential and magic numbers

- **Sign of SO?**
  \[
  V_{ls} \left( \frac{\ell \cdot s}{\hbar^2} \right) r_0^2 \frac{1}{r} \frac{d}{dr} f(r)
  \]
  \[V_{ls} = -0.44V\]

- **Consequence for**
  - \(0f_{\frac{7}{2}}\)
  - \(0g_{\frac{9}{2}}\)
  - \(0h_{\frac{11}{2}}\)
  - \(0i_{\frac{13}{2}}\)

- **Noticeably shifted**
- **Correct magic numbers!**

\[
\begin{align*}
N = 6, \pi^+ & \quad -0i, 1g, 2d, 3s^- \\
& \quad -0i_{\frac{13}{2}} \quad 2p_{\frac{1}{2}}^- \\
& \quad 1f_{\frac{7}{2}} \\
& \quad 0h_{\frac{9}{2}} \quad 2p_{\frac{3}{2}} \\
N = 5, \pi^- & \quad -0h, 1f, 2p^- \\
& \quad 1f_{\frac{7}{2}} \\
& \quad 0h_{\frac{9}{2}} \quad 2s_{\frac{1}{2}} \\
N = 4, \pi^+ & \quad -0g, 1d, 2s^- \\
& \quad 0g_{\frac{7}{2}} \quad 1d_{\frac{3}{2}} \\
& \quad 0g_{\frac{9}{2}} \quad 0s_{\frac{3}{2}} \\
N = 3, \pi^- & \quad -0f, 1p^- \\
& \quad 1p_{\frac{3}{2}} \\
& \quad 0f_{\frac{7}{2}} \quad 0p_{\frac{3}{2}} \\
N = 2, \pi^+ & \quad -0d, 1s^- \\
& \quad 1s_{\frac{1}{2}} \\
& \quad 0d_{\frac{3}{2}} \quad 0s_{\frac{1}{2}} \\
N = 1, \pi^- & \quad -0p^- \\
& \quad 0p_{\frac{3}{2}} \quad 0p_{\frac{5}{2}} \\
N = 0, \pi^+ & \quad -0s \\
& \quad 0s_{\frac{1}{2}}
\end{align*}
\]
\[ ^{208}\text{Pb for example} \]

- **Empirical potential & sp energies**
  \[ \hat{H}_0 \ a^\dagger_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle = \left[ \varepsilon_\alpha + E(208\text{Pb}_{g.s.}) \right] \ a^\dagger_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle \]

- **A+1:** “sp energies” \[ E^{A+1}_n - E^A_0 \] directly from experiment

- **A-1:**
  \[ \hat{H}_0 \ a_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle = \left[ E(208\text{Pb}_{g.s.}) - \varepsilon_\alpha \right] \ a_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle \]

- also directly from \[ E^A_0 - E^{A-1}_n \]

- Shell filling for nuclei near stability follows empirical potential
Comparison with experiment

• Now how to explain this potential ...
Nucleon-nucleon interaction

- Shell structure in nuclei and lots more to be explained on the basis of how nucleons interact with each other in free space

- QCD
- Lattice calculations
- Effective field theory
- Exchange of lowest bosonic states
- Phenomenology

- Realistic NN interactions: describe NN scattering data up to pion production threshold plus deuteron properties
- Note: extra energy scale from confinement of nucleons
Nuclear Matter

- Nuclear masses near stability

\[ M(N, Z) = \frac{E(N, Z)}{c^2} = N m_n + Z m_p - \frac{B(N, Z)}{c^2} \]

- Data

- Each A most stable N,Z pair

- Where fission?

- Where fusion?
Nuclear Matter

- Smooth curve

\[ B = b_{vol} A - b_{surf} A^{2/3} - \frac{1}{2} b_{sym} \frac{(N - Z)^2}{A} - \frac{3}{5} \frac{Z^2 e^2}{R_c} \]

- volume \hspace{1cm} b_{vol} = 15.56 \text{ MeV}

- surface \hspace{1cm} b_{surf} = 17.23 \text{ MeV}

- symmetry \hspace{1cm} b_{sym} = 46.57 \text{ MeV}

- Coulomb \hspace{1cm} R_c = 1.24 A^{1/3} \text{ fm}

Great interest in limit: \( N=Z; \) no Coulomb; \( A \rightarrow \infty \)

Two most important numbers in Nuclear Physics

\[ \frac{B}{A} \approx 16 \text{ MeV} \hspace{1cm} \rho_0 \approx 0.16 \text{ fm}^3 \]
Saturation problem of nuclear matter

Given $V_{NN} \Rightarrow$ explain correct minimum of $E/A$ in nuclear matter as a function of density inside empirical box

Describe the infinite system of neutrons

$\Rightarrow$ properties of neutron stars
Isospin

• Shell closures for N and Z the same!!
• Also $m_n c^2 \approx m_p c^2$ 939.56 MeV vs. 938.27 MeV
• So strong interaction Hamiltonian (QCD) invariant for $p \leftrightarrow n$
• But weak and electromagnetic interactions are not
• Strong interaction dominates $\Rightarrow$ consequences

• Notation (for now) $p^\dagger_\alpha$ adds proton $n^\dagger_\alpha$ adds neutron

• Anticommutation relations
  \[\{p^\dagger_\alpha, p_\beta\} = \delta_{\alpha,\beta}\]
  \[\{n^\dagger_\alpha, n_\beta\} = \delta_{\alpha,\beta}\]
Isospin

• Z proton & N neutron state

\[ |\alpha_1 \alpha_2 ... \alpha_Z; \beta_1 \beta_2 ... \beta_N \rangle = p_{\alpha_1}^{\dagger} p_{\alpha_2}^{\dagger} ... p_{\alpha_Z}^{\dagger} n_{\beta_1}^{\dagger} n_{\beta_2}^{\dagger} ... n_{\beta_N}^{\dagger} |0\rangle \]

• Exchange all p with n

\[ \hat{T}^+ = \sum_{\alpha} p_{\alpha}^{\dagger} n_{\alpha} \]

• and vice versa

\[ \hat{T}^- = \sum_{\alpha} n_{\alpha}^{\dagger} p_{\alpha} \]

• Expect

\[ [\hat{H}_S, \hat{T}^\pm] = 0 \]

• Consider

\[ \hat{T}_3 = \frac{1}{2} [\hat{T}^+, \hat{T}^-] = \frac{1}{2} \sum_{\alpha \beta} (p_{\alpha}^{\dagger} n_{\alpha} n_{\beta}^{\dagger} p_{\beta} - n_{\beta}^{\dagger} p_{\beta} p_{\alpha}^{\dagger} n_{\alpha}) \]

\[ = \frac{1}{2} \sum_{\alpha \beta} (p_{\alpha}^{\dagger} p_{\beta} \delta_{\alpha,\beta} - n_{\beta}^{\dagger} n_{\alpha} \delta_{\alpha,\beta}) = \frac{1}{2} \sum_{\alpha} (p_{\alpha}^{\dagger} p_{\alpha} - n_{\alpha}^{\dagger} n_{\alpha}) \]

• will also commute with \( H_S \)
Isospin

• Check \([\hat{T}_3, \hat{T}^\pm] = \pm \hat{T}^\pm\)

• Then operators

\[
\hat{T}_1 = \frac{1}{2} \left( \hat{T}^+ + \hat{T}^- \right)
\]
\[
\hat{T}_2 = \frac{1}{2i} \left( \hat{T}^+ - \hat{T}^- \right)
\]
\[
\hat{T}_3
\]

obey the same algebra as \(J_x, J_y, J_z\)

so spectrum identical and \(\hat{H}_S, \hat{T}^2, \hat{T}_3\) simultaneously diagonal!

proton

\[|r m_s\rangle_p = |r m_s m_t = \frac{1}{2}\rangle\]

neutron

\[|r m_s\rangle_n = |r m_s m_t = -\frac{1}{2}\rangle\]

For this doublet

\[T^2 |r m_s m_t\rangle = \frac{1}{2} \left( \frac{1}{2} + 1 \right) |r m_s m_t\rangle\]

and

\[T_3 |r m_s m_t\rangle = m_t |r m_s m_t\rangle\]

States with total isospin constructed as for angular momentum