Problem set 6

1.

In[1]:=
```
a = 100;
m = 0.1;
ℏ = 1;
```

$\psi_1[x\_] := \sqrt{\frac{2}{a}}\ \text{Cos}[\pi\, x / a];$

$\psi_2[x\_] := \sqrt{\frac{2}{a}}\ \text{Sin}[2\, \pi\, x / a];$

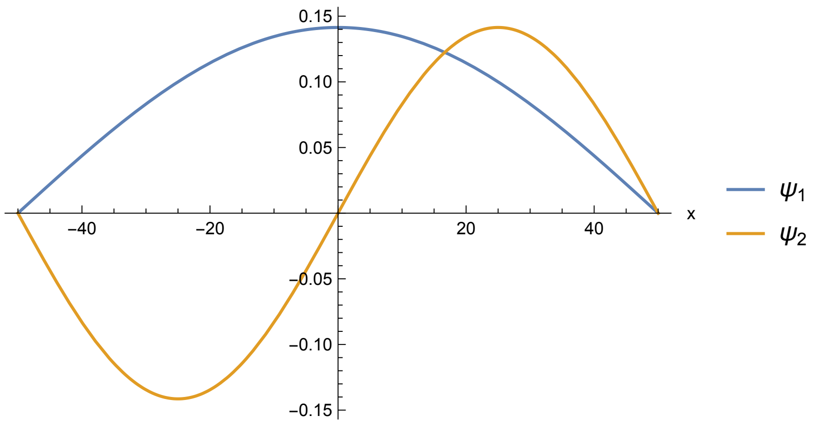
```
Plot[{ψ1[x], ψ2[x]}, {x, -50, 50}, AxesLabel → {"x"}, PlotLegends → {"ψ1", "ψ2"}]
```

$\int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_1{}^* \, x \, \psi_1 \, dx \qquad (* \ \langle\psi_1|\hat{x}|\psi_1\rangle \ *)$

$\int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_2{}^* \, x \, \psi_2 \, dx \qquad (* \ \langle\psi_2|\hat{x}|\psi_2\rangle \ *)$

```
(*  Alternatively, use symmetry arguments: since ψ*ψ is an even function,
and x is an odd function, the integration of (ψ*xψ),
which is an odd function, over the range (-a/2, a/2), is 0.  *)
```

Out[6]=



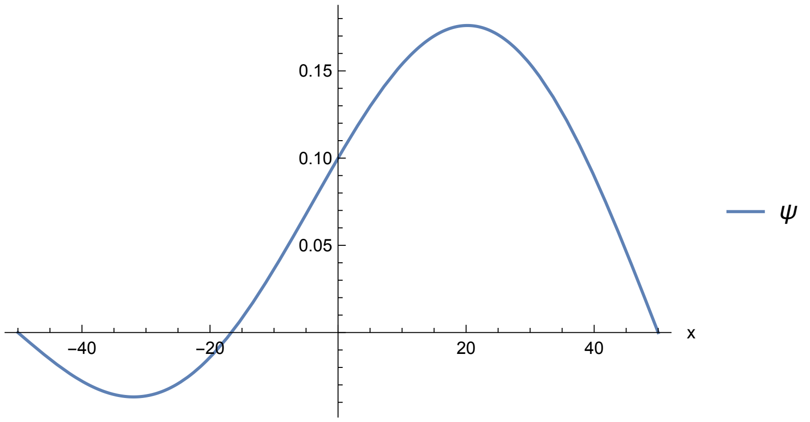
Out[7]= 0

Out[8]= 0

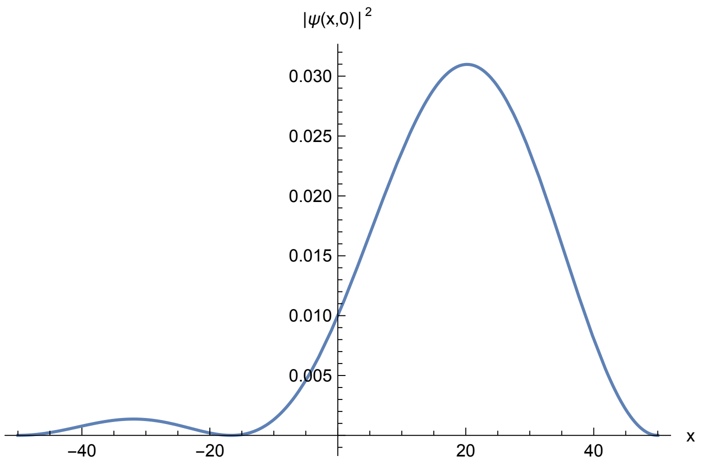
2. (a)

In[9]:=
```
wavefn$t0[x_] := 1/Sqrt[2] (ψ1[x] + ψ2[x]);
Plot[wavefn$t0[x], {x, -50, 50}, AxesLabel → {"x"},
 PlotLegends → {"ψ"}]          (*  Plot of ψ(x,0)   *)
Plot[Conjugate[wavefn$t0[x]] wavefn$t0[x],
 {x, -50, 50}, AxesLabel → {"x", "|ψ(x,0)|^2"}]
(*  Plot of the probability density |ψ(x,0)|^2   *)
(*  Since the probability density is higher around the region (-10, 40),
it is most likely to find the particle in that region at time t=0.   *)
```

Out[10]=



Out[11]=



(b)

In[12]:=
```
∫_{-a/2}^{a/2} Conjugate[wavefn$t0[x]] wavefn$t0[x] dx
(*  Check normalization of ψ(x,0)    *)
N[∫_{-a/2}^{a/2} Conjugate[wavefn$t0[x]] x wavefn$t0[x] dx]
(*  Expectation value of position. The value is around where
 the particle is most likely to find, which is resonable.   *)
```

Out[12]= 1

Out[13]= 18.0127

3. (a) The time dependence of the eigenfunction is $e^{-\frac{i E_n t}{\hbar}}$. Therefore, the angular frequency of the eigenfunciton is $\omega = \frac{E_n}{\hbar} = \frac{\pi^2 \hbar n^2}{2 m a^2}$. Hence, the angular frequency $\omega_1 = \frac{\pi^2 \hbar}{2 m a^2}$,

and for the second eigenfunction, the angular frequency $\omega_2 = \frac{2 \pi^2 \hbar}{m a^2}$.

The wavefunctions evolve in time :

$$\psi_1 (x, t) = \psi_1 (x)\, e^{-i \omega_1 t} = \sqrt{\frac{2}{a}}\, \text{Cos}[\pi x / a]\, e^{-i \frac{\pi^2 \hbar}{2 m a^2} t};$$

$$\psi_2 (x, t) = \psi_2 (x)\, e^{-i \omega_2 t} = \sqrt{\frac{2}{a}}\, \text{Sin}[2 \pi x / a]\, e^{-i \frac{2 \pi^2 \hbar}{m a^2} t}.$$

(b)

In[14]:=
```
wavefn[x_, t_] := 1/Sqrt[2] (ψ1[x] E^(-I π^2 ħ/(2 m a^2) t) + ψ2[x] E^(-I 2 π^2 ħ/(m a^2) t));
```

```
wavefn[x, 0]
wavefn$t0[x]          (*  Neglecting the 0.i term,
this gives the same wavefunction as wavefn$t0[x].  *)
```

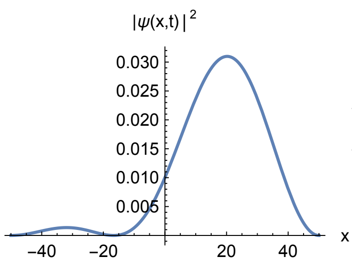
Out[15]= $\frac{(0.141421 + 0.\, i)\, \text{Cos}\left[\frac{\pi x}{100}\right] + (0.141421 + 0.\, i)\, \text{Sin}\left[\frac{\pi x}{50}\right]}{\sqrt{2}}$

Out[16]= $\frac{\frac{\text{Cos}\left[\frac{\pi x}{100}\right]}{5 \sqrt{2}} + \frac{\text{Sin}\left[\frac{\pi x}{50}\right]}{5 \sqrt{2}}}{\sqrt{2}}$

(c)

In[17]:=
```
DynamicModule[{t}, {Slider[Dynamic[t], {0, 600, 1}],
  "t =" Dynamic[t], Dynamic[Plot[Conjugate[wavefn[x, t]] wavefn[x, t],
    {x, -50, 50}, AxesLabel → {"x", "|ψ(x,t)|^2"}]]}]
```

Out[17]= { , t = 0,



}

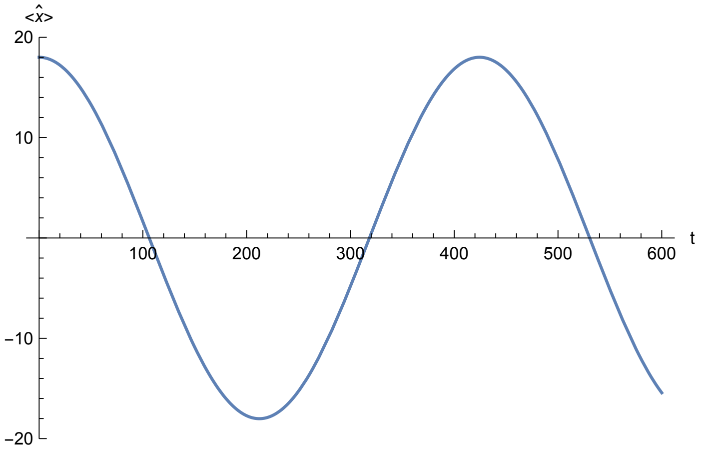
As time going on, the most probable position of the particle moves from around 20 to -20 gradually. Its period is roughly τ = 420. The particle roughly moves around (-20, 20), hence its amplitude of oscillations is about 20.

(d)

In[18]:=
```
expected$x[t_] := ∫_{-a/2}^{a/2} Conjugate[wavefn[x, t]] x wavefn[x, t] ⅆx;
expected$x[0]
(*  which is consistent with the last part of question 2  *)
Plot[expected$x[t], {t, 0, 600}, AxesLabel → {"t", "<x̂>"}]
```

Out[19]= 18.0127

Out[20]=



The particle moves back and forth about the origin. Its amplitude is around 18, and its period of oscillation is around (530 - 105 =) 425.

4.

$$< \psi \mid \hat{x} \mid \psi > = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{\sqrt{2}} \left(\psi_1 e^{-i \frac{\pi^2 \hbar}{2 m a^2} t} + \psi_2 e^{-i \frac{2 \pi^2 \hbar}{m a^2} t}\right)^* x \frac{1}{\sqrt{2}} \left(\psi_1 e^{-i \frac{\pi^2 \hbar}{2 m a^2} t} + \psi_2 e^{-i \frac{2 \pi^2 \hbar}{m a^2} t}\right) dx$$

$$= \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\psi_1^* x \psi_1 + \psi_2^* x \psi_2 + (\psi_1^* x \psi_2 + \psi_2^* x \psi_1) e^{i \left(\frac{\pi^2 \hbar}{2 m a^2} - \frac{2 \pi^2 \hbar}{m a^2}\right) t}\right) dx$$

(* We are already familiar with the result: $\int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_n^* \; x \; \psi_n dx = 0.$ *)

$$= \frac{1}{2} e^{i \left(\frac{\pi^2 \hbar}{2 m a^2} - \frac{2 \pi^2 \hbar}{m a^2}\right) t} \int_{-\frac{a}{2}}^{\frac{a}{2}} (\psi_1^* x \psi_2 + \psi_2^* x \psi_1) dx$$

$$= \frac{1}{2} e^{i \left(\frac{\pi^2 \hbar}{2 m a^2} - \frac{2 \pi^2 \hbar}{m a^2}\right) t} \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\cos\left(\frac{\pi}{a} x\right) x \sin\left(\frac{2 \pi}{a} x\right) + \sin\left(\frac{2 \pi}{a} x\right) x \cos\left(\frac{\pi}{a} x\right)\right) dx$$

(* Change variable $\frac{\pi}{a} x = \theta$ *)

$$= \frac{1}{a} e^{i \left(\frac{\pi^2 \hbar}{2 m a^2} - \frac{2 \pi^2 \hbar}{m a^2}\right) t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos(\theta) \frac{a}{\pi} \theta \sin(2 \theta) + \sin(2 \theta) \frac{a}{\pi} \theta \cos(\theta)\right) \frac{a}{\pi} d\theta$$

$$= \frac{1}{a} \frac{a^2}{\pi^2} e^{i \left(\frac{\pi^2 \hbar}{2 m a^2} - \frac{2 \pi^2 \hbar}{m a^2}\right) t} 2 \times \frac{8}{9}$$

$$= \frac{16 a}{9 \pi^2} e^{-\frac{4 m a^2}{3 \pi \hbar} i t}$$

(* The oscillation period is $\tau = \frac{4 \; m \; a^2}{3 \; \pi \; \hbar}$,
and the amplitude of the expected position is $\frac{16 \; a}{9 \; \pi^2}$. *)

$$= \frac{16 \times 100}{9 \pi^2} e^{-\frac{4 \times 0.1 \times 100^2}{3 \pi} i t}$$

$$= 18.0127 \, e^{-0.0148 \, i \, t}.$$

(* The amplitude of the oscillation of position expectation value is indeed 18.0127, and the period is $\tau = \frac{2 \; \pi}{0.0148} = 424.4$, which is about the same as our estimation. *)