1. At time $t = 0$, a particle moving in a simple harmonic motion has $x = 2\sqrt{3}$, $\dot{x} = 6$, and $\ddot{x} = -18\sqrt{3}$.

(a) Write an expression for the motion of the form $x = \text{Re}(\mathcal{A}e^{i\omega t})$, where $\mathcal{A}$ is a complex number.

SOLUTION - Using the expression given for $x(t)$ we can use the initial conditions to determine $\mathcal{A}$ and ω:

$$x(0) = \text{Re}(\mathcal{A}) = 2\sqrt{3} \tag{1}$$
$$\dot{x}(0) = \omega\text{Re}(i\mathcal{A}) = 6 \tag{2}$$
$$\ddot{x}(0) = -\omega^2\text{Re}(\mathcal{A}) = -18\sqrt{3} \tag{3}$$

Writing $\mathcal{A} = a+ib$, from equation 1 we have that $\mathcal{A} = 2\sqrt{3}+ib$. With this information, we can find ω directly from equation 3,

$$2\sqrt{3}\omega^2 = 18\sqrt{3} \rightarrow \omega = 3. \tag{4}$$

Now, pluggling these values to equation 2 we have

$$\omega\text{Re}(i\mathcal{A}) = 6 \rightarrow \text{Re}(i\mathcal{A}) = 2 \tag{5}$$
$$i\mathcal{A} = ia - b \rightarrow \text{Re}(i\mathcal{A}) = -b = 2 \tag{6}$$

Putting all together we have $x(t) = \text{Re}[(2\sqrt{3} - 2i)e^{3it}]$ $\square$

(b) Write an expression for the motion of the form $x = \mathcal{A}\cos(\omega t - \phi)$, where $\mathcal{A}$ and ϕ are real.

SOLUTION - Using the same approach we find

$$x(0) = \mathcal{A}\cos\phi = 2\sqrt{3} \tag{7}$$
$$\dot{x}(0) = \omega\mathcal{A}\sin\phi = 6 \tag{8}$$
$$\ddot{x}(0) = -\omega\mathcal{A}\cos\phi = -18\sqrt{3} \tag{9}$$

Dividing equation 7 by 9 we find that $\omega = 3$, as before. From equation 7 and 8 we have

$$\mathcal{A}\sin\phi = 2 \quad \text{and} \quad \mathcal{A}\cos\phi = 2\sqrt{3} \tag{10}$$
$$\mathcal{A}^2(\sin^2\phi + \cos^2\phi) = 16 \rightarrow \mathcal{A} = 4 \tag{11}$$

Lastly we can determine ϕ:

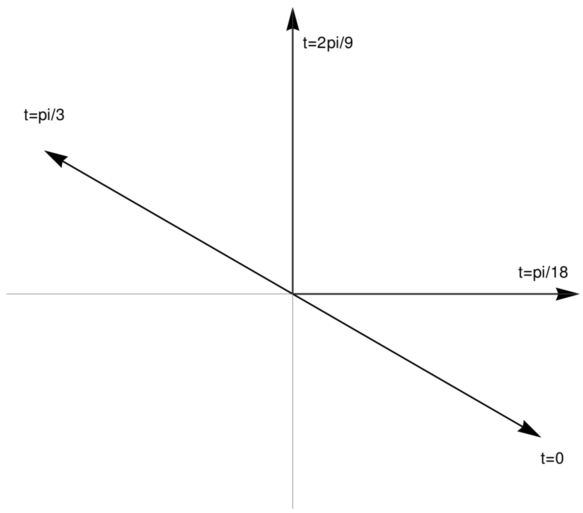
$$12\sin\phi = 6 \rightarrow \phi = \pi/6 \tag{12}$$

adn write out $x(t) = 4\cos(3t - \pi/6)$ $\square$.

(c) Represent $\mathcal{A}\exp^{i\omega t}$ as a rotating vector in the complex plane. Draw a diagram showing its position at $t = 0$, $t = \pi/18$, $t = 2\pi/9$, and $t = \pi/3$.

SOLUTION - We can represent $x(t) = 4\cos(3t - \pi/6)$ in a complex plane using its magnitude and corresponding angle as follows:

$$x(0) = 4\cos(-\pi/6) = 2\sqrt{3} \tag{13}$$
$$x(\pi/18) = 4\cos(0) = 4 \tag{14}$$
$$x(2\pi/9) = 4\cos(3\pi/6) = 0 \tag{15}$$
$$x(\pi/3) = 4\cos(5\pi/6) = -2\sqrt{3} \tag{16}$$



2. A particle of mass m undergoes damped oscillations with damping coefficient β and natural frequency $\omega_0 (\omega_0 \gg \beta)$. At $t = 0$, it starts at $x = A$ with $\dot{x} = 0$.

(a) Calculate the kinectic energy, potential energy, and total energy as functions of time.

SOLUTION - Starting from a general solution for damped oscillations $x(t) = \mathcal{C}e^{-\beta t}\cos(\omega_1 t - \delta)$, we apply the initial conditions to determine the constants $\mathcal{C}$ and δ:

$$x(0) = \mathcal{A} \rightarrow \cos\delta = \mathcal{A}/\mathcal{C} \tag{17}$$
$$\dot{x}(0) = 0 \rightarrow \sin\delta = \frac{\beta}{\omega_1}\cos\delta = \frac{\beta\mathcal{A}}{\omega_1\mathcal{C}} \tag{18}$$

Using that $\cos\delta^2 + \sin\delta^2 = 1$, we find that

$$\mathcal{C} = \mathcal{A}\sqrt{1 + \left(\frac{\beta}{\omega_1}\right)^2} \tag{19}$$

and because this is a damped osclliation, we have $\omega_0 \gg \beta$, which implies that $\omega_1 \approx \omega_0$ and the expression for $\mathcal{C}$ can be simplified to just being equal to $\mathcal{A}$. Therefore this system can be described in terms of

$$x(t) = \mathcal{A}e^{-\beta t}\cos(\omega_1 t - \delta). \tag{20}$$

With this expression it's possible to calculate its potential and kinectic energy:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2\mathcal{A}^2e^{-2\beta t}\cos^2(\omega_1 t - \delta) \tag{21}$$

$$\begin{aligned} T &= \frac{1}{2}m\dot{x} = \frac{1}{2}m\mathcal{A}^2e^{-2\beta t}[\beta\cos(\omega_1 t - \delta) + \omega_1\sin(\omega_1 t - \delta)]^2 \\ &= \frac{1}{2}m\mathcal{A}^2e^{-2\beta t}[\beta^2\cos^2(\omega_1 t - \delta) + \omega_1^2\sin^2(\omega_1 t - \delta) + 2\beta\omega_1\cos(\omega_1 t - \delta)\sin(\omega_1 t - \delta)] \end{aligned} \tag{22}$$

and total energy E, where three expressions were used to obtain its final form: $\omega_1^2 = \omega_0^2 - \beta^2$, $\cos^2\theta + \sin^2\theta = 1$ and $\sin(2\theta) = 2\cos\theta\sin\theta$:

$$E = \frac{1}{2}m\mathcal{A}^2e^{-2\beta t}\{\omega_0^2 + \beta^2[\cos^2(\omega_1 t - \delta) - \sin^2(\omega_1 t - \delta)] + \beta\omega_1\sin(\omega_1 t - \delta)\} \quad \square \tag{23}$$

(b) What is the average total energy (average over one cycle)? [Hint: Since $\omega_0 \gg \beta$, one may assume that $\exp^{-\beta t}$ stays relatively constant over one cycle. Then the average energy can be found by averaging only those terms that contain $\omega_1 t$. Answer: $E \approx \frac{1}{2}m\omega_0 A^2\exp^{-2\beta t}$.]

SOLUTION - Averaging terms with ωt - we can ignore δ because it's a constant phase shift and it won't change the average value. Starting with $\cos^2(\omega t)$:

$$\langle\cos^2(\omega t)\rangle = \left\langle\frac{1+\cos(\omega t)}{2}\right\rangle = \frac{1}{2} + \frac{1}{2}\left[\frac{1}{T}\int_0^T \cos(\omega t)dt\right] \tag{24}$$

$$= \frac{1}{2} + \frac{1}{2}\left[\frac{1}{T}\sin(\omega t)\Big|_0^{2\pi/\omega}\right] = \frac{1}{2} \tag{25}$$

and the same result can be calculated for $\langle\sin^2(\omega t)\rangle = 1/2$. Given that the integral of a periodic function over its period is zero, we have that $\langle\sin(2\omega t)\rangle = 0$; then we can write that the average total energy of this system is

$$\langle E\rangle = \frac{1}{2}m\mathcal{A}^2e^{-2\beta t}\left[\omega_0^2 + \beta^2(\frac{1}{2} - \frac{1}{2}) + \beta\omega_1(0)\right] = \frac{1}{2}m\omega_0^2\mathcal{A}^2e^{-2\beta t} \tag{26}$$

3. When a body is suspended from a fized point by a certain linear spring (i.e. obeying Hooke's law), the natural frequency of its vertical oscillations is found to be ω_1. When a different linear spring is used, the oscillations have angular frequency ω_2.

(a) Find the angular frequency when the two springs are used together in parallel.

SOLUTION - In this case the displacement is the same fo both springs, thus

$$F = -k_1 x_1 - k_2 x_2 = -\underbrace{(k_1 + k_2)}_{\text{k-parallel}} x \tag{27}$$

With this expression for $k_parallel$ we can write $\omega_{parallel}$

$$\omega_{parallel} = \sqrt{\frac{k_{parallel}}{m}} = \sqrt{\frac{k_1 + k_2}{m}} \tag{28}$$

$$\omega_{parallel} = \sqrt{\omega_1^2 + \omega_2^2} \quad \square \tag{29}$$

(b) Repeat the calculation when they are used in series.

SOLUTION - In this case the total displacement is the sum of the displacements of each spring

$$F = -k_{series}(x_1 + x_2) \tag{30}$$

Because the force on each spring has to be equal, we have

$$F_{12} = F_{21} \rightarrow -k_1 x_1 = -k_2 x_2 \rightarrow x_1 = \frac{k_2}{k_1} x_2 \tag{31}$$

$$F = F_{21} \rightarrow -k_{series}\left(\frac{k_2}{k_1} + 1\right) = -k_2 x_2 \tag{32}$$

$$k_{series} = \frac{k_1 k_2}{k_1 + k_2} \tag{33}$$

Therefore we have

$$\omega_{series} = \sqrt{\frac{k_{series}}{m}} = \frac{\omega_1 \omega_2}{\sqrt{\omega_1^2 + \omega_2^2}} \quad \square \tag{34}$$

(c) Show that the first of these frequencies is at least twice the second.

SOLUTION - The ratio between $\omega_{parallel}$ and ω_{series} is

$$\frac{\omega_p}{\omega_s} = \frac{\omega_1^2 + \omega_2^2}{\omega_1 \omega_2} + \underbrace{\frac{2\omega_1\omega_2}{\omega_1\omega_2} - \frac{2\omega_1\omega_2}{\omega_1\omega_2}}_{\text{completing the square}} = \frac{(\omega_1^2 - \omega_2^2)^2}{\omega_1 \omega_2} + 2 \tag{35}$$

Since $\omega_1 \omega_2$ is always greater than zero, we can see that ω_p/ω_s has to be always greater than 2.

4. The position of an overdamped harmonic oscillator is given by Eq. (5.40) in the text.

(a) Find the constants C_1 and C_2 in terms of the initial position x_0 and velocity v_0.

SOLUTION - We use the general solution for an overdamped harmonic oscillator, given by

$$x(t) = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t} \tag{36}$$

From the initial conditions,

$$C_1 + C_2 = x_0 \rightarrow C_2 = x_0 - C - 1 \tag{37}$$

$$-(\beta - \sqrt{\beta^2 - \omega_0^2})C_1 - (\beta + \sqrt{\beta^2 - \omega_0^2})C_2 = v_0 \tag{38}$$

From equation 37:

$$2\sqrt{\beta^2 - \omega_0^2}C_1 - \left(\beta + \sqrt{\beta^2 - \omega_0^2}\right) = v_0 \tag{39}$$

$$C_1 = \frac{v_0 + (\beta + \sqrt{\beta^2 - \omega_0^2})x_0}{2\sqrt{\beta^2 - \omega_0^2}} \tag{40}$$

and therefore

$$C_2 = x_0 - \left[\frac{v_0 + (\beta + \sqrt{\beta^2 - \omega_0^2})x_0}{2\sqrt{\beta^2 - \omega_0^2}}\right] = -\frac{(\beta - \sqrt{\beta^2 - \omega_0^2})x_0 - v_0}{2\sqrt{\beta^2 - \omega_0^2}} \quad \square \tag{41}$$

(b) Plot the resulting $x(t)$ for the two cases that $v_0 = 0$ and $x_0 = 0$.

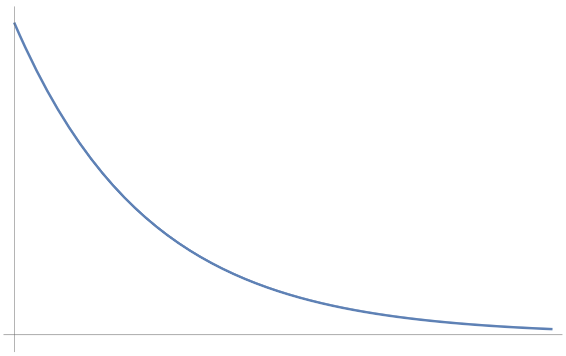
SOLUTION - For $v_0 = 0$, we have

$$x(t) = \frac{\beta + \sqrt{\beta^2 - \omega_0^2}}{2\sqrt{\beta^2 - \omega_0^2}} x_0 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} - \frac{(\beta - \sqrt{\beta^2 - \omega_0^2})x_0}{2\sqrt{\beta^2 - \omega_0^2}} e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t} \tag{42}$$

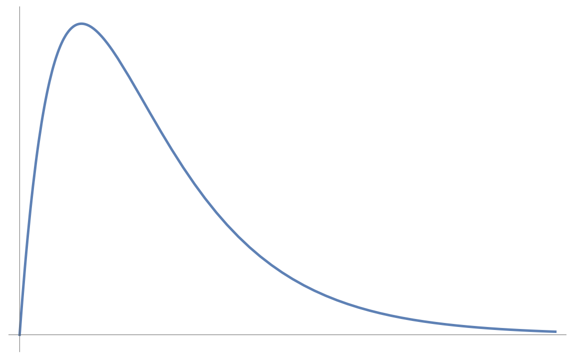
Meaning that after it's released, the overdamped HO returns to its equilibrium position as $t \rightarrow \infty$.
For the other case where $x_0 = 0$,

$$x(t) = \frac{v_0}{2\sqrt{\beta^2 - \omega_0^2}} e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} - \frac{v_0}{2\sqrt{\beta^2 - \omega_0^2}} e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})} \tag{43}$$

This time, after the initial "kick" the system moves to a maximum displacement and then returns to its equilibruim position as $t \rightarrow \infty$.



$v_0 = 0$



$x_0 = 0$

(c) Show that for $\beta \to 0$ your solution in a) approaches the solution for undamped motion.

SOLUTION - Starting with the original expression for $x(t)$:

$$\lim_{\beta \to 0} = C_1 e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + C_2 e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t} \approx C_1 e^{\sqrt{-\omega_0^2} t} + C_2 e^{-\sqrt{\omega_0^2} t} \tag{44}$$

$$x(t) = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \tag{45}$$

which is the solution for the undamped harmonic oscillator. Also, for $\beta \to 0$, $C_1 = C_2*$ so we can write

$$x(t) = \frac{x_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) + \frac{v_0}{2i\omega}\left(e^{i\omega t} - e^{-i\omega t}\right) \tag{46}$$

$$= x_0 \cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \quad \square \tag{47}$$