1. Suppose it is known about a vector $\mathbf{A}$ only that its projection on a given unit vector $\hat{\mathbf{e}}$ is $s = \hat{\mathbf{e}} \cdot \mathbf{A}$, and that $\hat{\mathbf{e}} \times \mathbf{A} = \mathbf{B}$. Show that this is sufficient to determine $\mathbf{A}$ by expressing it in terms of $\hat{\mathbf{e}}$, s, and $\mathbf{B}$. [Hint: Resolve $\mathbf{A}$ into its components along the three perpendicular axes given by $\hat{\mathbf{e}}$, $\mathbf{B}/|\mathbf{B}|$ and $\hat{\mathbf{e}} \times \mathbf{B}/|\mathbf{B}|$].

SOLUTION - Decomposing $\mathbf{A}$ into its components we get

$$\mathbf{A} = A_1\hat{\mathbf{e}} + A_2\frac{\mathbf{B}}{|\mathbf{B}|} + A_3\frac{\hat{\mathbf{e}} \times \mathbf{B}}{|\mathbf{B}|} \tag{1}$$

and with the information provided by the problem we can solve for A_1, A_2 and A_3. First we have that $\hat{\mathbf{e}} \cdot \mathbf{A} = s$, which means $A_1 = s$. Then we use $\hat{\mathbf{e}} \times \mathbf{A} = \mathbf{B}$ to find the other two components

$$\hat{\mathbf{e}} \times \mathbf{A} = \mathbf{B} \tag{2}$$

$$A_2\frac{\hat{\mathbf{e}} \times \mathbf{B}}{|\mathbf{B}|} + A_3\frac{\hat{\mathbf{e}} \times (\hat{\mathbf{e}} \times \mathbf{B})}{|\mathbf{B}|} = \mathbf{B} \tag{3}$$

Using the triple product identity, we can rewrite the second term on the left hand side as

$$\hat{\mathbf{e}} \times (\hat{\mathbf{e}} \times \mathbf{B}) = \hat{\mathbf{e}} \cdot \cancelto{0}{(\hat{\mathbf{e}} \cdot \mathbf{B})} - \mathbf{B} \cdot (\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}) \tag{4}$$

$$= -\mathbf{B} \tag{5}$$

Back to equation 2, we find the values of A_2 and A_3 by direct comparison of the two sides

$$A_2\frac{\hat{\mathbf{e}} \times \mathbf{B}}{|\mathbf{B}|} - A_3\frac{-\mathbf{B}}{|\mathbf{B}|} = \mathbf{B} \tag{6}$$

$$A_2 = 0 \qquad \text{and} \qquad A_3 = -|\mathbf{B}| \tag{7}$$

Thus

$$\mathbf{A} = s\hat{\mathbf{e}} - \hat{\mathbf{e}} \times \mathbf{B} \quad \square$$

2. Prove the following identities

(a)

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{8}$$

SOLUTION - Writing **A**, **B** and **C** components explicitly we have

$$\mathbf{A} = (A_1, A_2, A_3) \tag{9}$$
$$\mathbf{B} = (B_1, B_2, B_3) \tag{10}$$
$$\mathbf{C} = (C_1, C_2, C_3). \tag{11}$$

From the first term of equation 8, the cross product between **A** and **B** is given by

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\hat{\mathbf{x}} + (A_1B_3 - A_3B_1)\hat{\mathbf{y}} + (A_1B_2 - A_2B_1)\hat{\mathbf{z}} \tag{12}$$

And it follows that

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1 \tag{13}$$

Rearranging these terms we get

$$\underbrace{A_1(B_2C_3 - B_3C - 2)}_{\mathbf{A}_{\hat{\mathbf{x}}}(\mathbf{B}\times\mathbf{C})_{\hat{\mathbf{x}}}} + \underbrace{A_2(B_3C_1 - B_1C - 3)}_{\mathbf{A}_{\hat{\mathbf{y}}}(\mathbf{B}\times\mathbf{C})_{\hat{\mathbf{y}}}} + \underbrace{A_3(B_1C_2 - B_2C_1)}_{\mathbf{A}_{\hat{\mathbf{z}}}(\mathbf{B}\times\mathbf{C})_{\hat{\mathbf{z}}}} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{14}$$

Alternatively from a different combination of the same terms

$$\underbrace{B_1(A_3C_2 - A_2C_3)}_{\mathbf{B}_{\hat{\mathbf{x}}}(\mathbf{C}\times\mathbf{A})_{\hat{\mathbf{x}}}} + \underbrace{B_2(A_1C_3 - A_3C_1)}_{\mathbf{B}_{\hat{\mathbf{y}}}(\mathbf{C}\times\mathbf{A})_{\hat{\mathbf{y}}}} + \underbrace{B_3(A_2C_1 - A_1C_2)}_{\mathbf{B}_{\hat{\mathbf{z}}}(\mathbf{C}\times\mathbf{A})_{\hat{\mathbf{z}}}} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \quad \square \tag{15}$$

(b) Note the cyclic order of the individual vectors.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{16}$$

SOLUTION - We can focus on only one compononent of the triple product since the procedure is similar for all three components. Starting from the expression for $\mathbf{B} \times \mathbf{C}$

$$\mathbf{B} \times \mathbf{C} = (B_2C_3 - B_3C_2)\hat{\mathbf{x}} + (B_3C_1 - B_3C_1)\hat{\mathbf{y}} + (B_1C_2 - B_2C_1)\hat{\mathbf{z}} \tag{17}$$

And

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = [A_2(B_1C_2 - B_2C_1) - A_3(B_3C_1 - B_1C_3)]\hat{\mathbf{x}} \\ +[A_3(B_2C_3 - B_3C_2) - A_1(B_1C_2 - B_2C_1)]\hat{\mathbf{y}} \\ +[A_1(B_3C_1 - B_1C_3) - A_2(B_2C_3 - B_3C_2)]\hat{\mathbf{z}} \tag{18}$$

which is equivalent to

$$= [B_1(A_2C_2 - A_3C_3) - C_1(A_2B_2 - A_3B_3)]\hat{\mathbf{x}} \\ +[B_2(A_1C_1 - A_3C_3) - C_2(A_1B_1 - A_3B_3)]\hat{\mathbf{y}} \\ +[B_3(A_1C_1 - A_2C_2) - C_3(A_1B_1 - A_2B_2)]\hat{\mathbf{z}} \tag{19}$$

Adding $B_iA_iC_i - C_iA_iB_i (= 0)$ to each component, in this case x direction we have

$$\hat{\mathbf{x}} : B_1(A_2C_2 + A_3C_3) + B_1A_1C_1 - C_1(A_2B_2 + A_3B_3) - C_1A_1B_1$$

$$= B_1(A_1C_1 + A_2C_2 + A_3C_3) - C_1(A_1B_1 + A_2B_2 + A_3B_3) \tag{20}$$

$$= [\mathbf{B}(\mathbf{A}\cdot\mathbf{C})]_{\hat{\mathbf{x}}} - [\mathbf{C}(\mathbf{A}\cdot\mathbf{B})]_{\hat{\mathbf{x}}} \tag{21}$$

By analogy the same result can be obtained for y and z components, completing this proof. □

3. David and Goliath are pushing a frictionless automobile of mass m which is initially at rest. Goliath, strong but out of shape, applies a forward push of magnitude $F(t) = F_0 \exp^{t/\tau}$, where τ and F_0 are constants. David, weaker but with more endurance, applies a backward push of magnitude $F(t) = \frac{1}{2}F_0 \exp^{-t/2\tau}$. What net displacement do they produce as $t \to \infty$?

SOLUTION - To find the net displacement we start with the net force applied to the automobile

$$F(t) = m \cdot a(t) = F_0(\exp^{-t/\tau} - 1/2 \exp^{-t/2\tau}) \tag{22}$$

Integrating twice over t on both sides we first get the velocity

$$v(t) = \frac{F_0}{m}\int_0^T \left(\exp^{-t/\tau} - \frac{1}{2}\exp^{-t/2\tau}\right) dt \\ = \frac{F_0\tau}{m}(-\exp^{-t/\tau} + \exp^{-t/2\tau}) \tag{23}$$

and finally position as functions of time

$$
\begin{aligned}
x(t) &= \frac{F_0\tau}{m}\int_0^T (\exp^{-t/2\tau} - \exp^{-t/\tau})dt \\
&= \frac{F_0\tau}{m}[-2\tau(\exp^{-T/2\tau} - 1) - \tau(1 - \exp^{-T/\tau})] \\
&= \frac{F_0\tau^2}{m}[1 - 2\exp^{T/2\tau} + \exp^{-T/\tau}]
\end{aligned} \tag{24}
$$

Given that the expression above is valid for any T, we can take the limit as $t \to \infty$ which gives

$$
\lim_{t\to\infty} x(t) = \frac{F_0\tau^2}{m} \quad \square \tag{25}
$$

4. A child stands on a scale in an elevator. In each of the following cases, explain whether the scale reads an amount which is less than, equal to, or greater than the childs weight. Discuss from the point of view of someone in an inertial frame, i.e., a frame which is not accelerating.

(a) The elevator speeds up as it leaves floor 1 for floor 3
SOLUTION - Scale measure the normal force acting on the child, which means that in this situation where there's a net force pointing up - resulting in a bigger response from the normal force - it will measure her weight being greater then the gravitaional force. $w > mg$

(b) The elevator speeds up as it leaves floor 3 for floor 1
SOLUTION - This is the opposite situation, where the net force points downwards. Here the scale measures her weight as less than the gravitational force. $w < mg$

(c) The elevator slows down as it approaches floor 1 from floor 3
SOLUTION - Although the elevator is going down, the net force in this situation is pointing upwards, giving the same result as in part a). $w > mg$

(d) The elevator moves at constant speed going from floor 1 to floor 3
SOLUTION - There is no net force being applied to the child, to normal and gravitational forces are equal. $w = mg$