Quantum Theory of Many-Particle Systems, Phys. 540

- IPM?
- Atoms?
- Nuclei: more now
- Other questions about last class?
- Assignment for next week Wednesday --->
- Comments?
Nuclear shell structure

• Ground-state spins and parity of odd nuclei provide further evidence of “magic numbers”

• Character of magic numbers may change far from stability (hot)


• $N=20$ may disappear and $N=16$ may appear
Empirical potential

- Analogy to atoms suggests finding a sp potential $\Rightarrow$ shells + IPM
- Difference(s) with atoms?
- Properties of empirical potential
  - overall?
  - size?
  - shape?
- Consider nuclear charge density

Nuclear density distribution

- Central density \((A/Z^*\) charge density) about the same for nuclei heavier than \(^{16}\text{O}\), corresponding to 0.16 nucleons/fm\(^3\)
- Important quantity
- Shape roughly represented by
  \[
  \rho_{ch}(r) = \frac{\rho_0}{1 + \exp\left(\frac{r-c}{z}\right)}
  \]
  \[
  c \approx 1.07A^{1/3}\text{fm}
  \]
  \[
  z \approx 0.55\text{fm}
  \]
- Potential similar shape
Empirical potential

• BM1

\[ U = V f(r) + V_{\ell s} \left( \frac{\ell \cdot s}{\hbar^2} \right) r_0^2 \frac{1}{r} \frac{d}{dr} f(r) \]

• Central part roughly follows shape of density

\[ f(r) = \left[ 1 + \exp \left( \frac{r - R}{a} \right) \right]^{-1} \]

• Woods-Saxon form

• Depth

\[ V = \left[ -51 \pm 33 \left( \frac{N - Z}{A} \right) \right] \text{ MeV} \]

  + neutrons

  - protons

• radius

\[ R = r_0 A^{1/3} \text{ with } r_0 = 1.27 \text{ fm} \]

• diffuseness

\[ a = 0.67 \text{ fm} \]
Analytically solvable alternative

- Woods-Saxon (WS) generates finite number of bound states
- IPM: fill lowest levels $\Rightarrow$ nuclear shells $\Rightarrow$ magic numbers
- reasonably approximated by 3D harmonic oscillator

\[ U_{HO}(r) = \frac{1}{2}m\omega^2 r^2 - V_0 \]

\[ H_0 = \frac{p^2}{2m} + U_{HO}(r) \]

- Eigenstates in spherical basis

\[ H_{HO} |n\ell m_\ell m_s\rangle = \left( \hbar \omega (2n + \ell + \frac{3}{2}) - V_0 \right) |n\ell m_\ell m_s\rangle \]
Harmonic oscillator

- Filling of oscillator shells
- # of quanta \( N = 2n + \ell \)

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<th>( n )</th>
<th>( \ell )</th>
<th># of particles</th>
<th>“magic #”</th>
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Need for another type of sp potential

- 1949 Mayer and Jensen suggest the need of a spin-orbit term
- Requires a coupled basis

\[
|n(\ell s)jm_j\rangle = \sum_{m_{\ell}m_s} |n\ell m_{\ell}m_s\rangle (\ell m_{\ell} s m_s | j m_j) 
\]

- Use \( \ell \cdot s = \frac{1}{2} (j^2 - \ell^2 - s^2) \) to show that these are eigenstates

\[
\frac{\ell \cdot s}{\hbar^2} |n(\ell s)jm_j\rangle = \frac{1}{2} \left( j(j + 1) - \ell(\ell + 1) - \frac{1}{2} \left( \frac{1}{2} + 1 \right) \right) |n(\ell s)jm_j\rangle
\]

- For \( j = \ell + \frac{1}{2} \) eigenvalue \( \frac{1}{2} \ell \)
- while for \( j = \ell - \frac{1}{2} \) \(-\frac{1}{2} (\ell + 1)\)
- so SO splits these levels! and more so with larger \( \ell \)
Inclusion of SO potential and magic numbers

- **Sign of SO?**
  \[ V_{\ell s} \left( \frac{\ell \cdot s}{\hbar^2} \right) r_0^2 \frac{1}{r} \frac{d}{dr} f(r) \]
  \[ V_{\ell s} = -0.44V \]

- **Consequence for**
  \[ 0f_{\frac{7}{2}} \]
  \[ 0g_{\frac{9}{2}} \]
  \[ 0h_{\frac{11}{2}} \]
  \[ 0i_{\frac{13}{2}} \]

- **Noticeably shifted**

- **Correct magic numbers!**
\[ ^{208}\text{Pb} \text{ for example} \]

- Empirical potential & sp energies
  \[ \hat{H}_0 \ a^\dagger_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle = [\varepsilon_\alpha + E(^{208}\text{Pb}_{g.s.})] \ a^\dagger_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle \]

- \( A+1 \): “sp energies” \( E_{n}^{A+1} - E_{0}^{A} \) directly from experiment
- \( A-1 \):
  \[ \hat{H}_0 \ a_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle = [E(^{208}\text{Pb}_{g.s.}) - \varepsilon_\alpha] \ a_\alpha \ |^{208}\text{Pb}_{g.s.}\rangle \]

- also directly from \( E_{0}^{A} - E_{n}^{A-1} \)

- Shell filling for nuclei near stability follows empirical potential
Comparison with experiment

• Now how to explain this potential ...
Closed-shells and angular momentum

- Atoms: consider one closed shell (argument the same for more)

\[ |n\ell m_\ell = \ell m_s = \frac{1}{2}, n\ell m_\ell = \ell m_s = -\frac{1}{2}, \ldots n\ell m_\ell = -\ell m_s = \frac{1}{2}, n\ell m_\ell = -\ell m_s = -\frac{1}{2} \rangle \]

- Expect?

- Example: He

\[
| (1s)^2 \rangle = \frac{1}{\sqrt{2}} \left\{ |1s \uparrow 1s \downarrow \rangle - |1s \downarrow 1s \uparrow \rangle \right\} \\
= | (1s)^2 ; L = 0 S = 0 \rangle
\]

- Consider nuclear closed shell

\[ | \Phi_0 \rangle = | n(\frac{1}{2}) j m_j = j, n(\frac{1}{2}) j m_j = j - 1, \ldots, n(\frac{1}{2}) m_j = -j \rangle \]
Angular momentum and second quantization

- **z-component of total angular momentum**
  \[
  \hat{J}_z = \sum_{n \ell j m} \sum_{n' \ell' j' m'} \langle n \ell j m | j_z | n' \ell' j' m' \rangle a_{n \ell j m}^\dagger a_{n' \ell' j' m'}
  \]
  \[
  = \sum_{n \ell j m} \hbar m \ a_{n \ell j m}^\dagger a_{n \ell j m}
  \]

- **Action on single closed shell**
  \[
  \hat{J}_z |n \ell j; m = -j, -j + 1, \ldots, j\rangle = \sum_{m} \hbar m \ a_{n \ell j m}^\dagger a_{n \ell j m} |n \ell j; m = -j, -j + 1, \ldots, j\rangle
  \]
  \[
  = \{ \sum_{m=-j}^{j} \hbar m \} |n \ell j; m = -j, -j + 1, \ldots, j\rangle
  \]
  \[
  = 0 \times |n \ell j; m = -j, -j + 1, \ldots, j\rangle
  \]

- **Also** \( \hat{J}_\pm |n \ell j; m = -j, -j + 1, \ldots, j\rangle = 0 \)

- **So total angular momentum** \( J = 0 \)

- **Closed shell atoms**
  \( L = 0 \)
  \( S = 0 \)
Nucleon-nucleon interaction

• Shell structure in nuclei and lots more to be explained on the basis of how nucleons interact with each other in free space

• QCD
• Lattice calculations
• Effective field theory
• Exchange of lowest bosonic states
• Phenomenology

• Realistic NN interactions: describe NN scattering data up to pion production threshold plus deuteron properties
• Note: extra energy scale from confinement of nucleons
Nuclear Matter

- Nuclear masses near stability
  \[ M(N, Z) = \frac{E(N, Z)}{c^2} = N m_n + Z m_p - \frac{B(N, Z)}{c^2} \]

- Data

- Each \( A \) most stable \( N, Z \) pair

- Where fission?

- Where fusion?
Nuclear Matter

- Smooth curve

$$B = b_{vol} A - b_{surf} A^{2/3} - \frac{1}{2} b_{sym} \frac{(N - Z)^2}{A} - \frac{3}{5} \frac{Z^2 e^2}{R_c}$$

- Volume
  $$b_{vol} = 15.56 \text{ MeV}$$

- Surface
  $$b_{surf} = 17.23 \text{ MeV}$$

- Symmetry
  $$b_{sym} = 46.57 \text{ MeV}$$

- Coulomb
  $$R_c = 1.24 A^{1/3} \text{ fm}$$

Great interest in limit: $N=Z$; no Coulomb; $A \to \infty$

Two most important numbers in Nuclear Physics

$$\frac{B}{A} \approx 16 \text{ MeV} \quad \rho_0 \approx 0.16 \text{ fm}^3$$
Saturation problem of nuclear matter

Given $V_{NN} \Rightarrow$ predict correct minimum of $E/A$ in nuclear matter as a function of density inside empirical box

Describe the infinite system of neutrons

$\Rightarrow$ properties of neutron stars
Isospin

- Shell closures for N and Z the same!!
- Also \( m_n c^2 \approx m_p c^2 \) 939.56 MeV vs. 938.27 MeV
- So strong interaction Hamiltonian (QCD) invariant for p ⇔ n
- But weak and electromagnetic interactions are not
- Strong interaction dominates ⇔ consequences

- Notation (for now) \( p^\dagger_\alpha \) adds proton
  \( n^\dagger_\alpha \) adds neutron

- Anticommutation relations \( \{ p^\dagger_\alpha, p_\beta \} = \delta_\alpha,\beta \)
  \( \{ n^\dagger_\alpha, n_\beta \} = \delta_\alpha,\beta \)
- All others 0
Isospin

- Z proton & N neutron state
  \[ |\alpha_1 \alpha_2 \ldots \alpha_Z; \beta_1 \beta_2 \ldots \beta_N \rangle = p_{\alpha_1}^\dagger p_{\alpha_2}^\dagger \ldots p_{\alpha_Z}^\dagger n_{\beta_1}^\dagger n_{\beta_2}^\dagger \ldots n_{\beta_N}^\dagger |0\rangle \]

- Exchange all p with n
  \[ \hat{T}^+ = \sum_{\alpha} p_{\alpha}^\dagger n_{\alpha} \]

- and vice versa
  \[ \hat{T}^- = \sum_{\alpha} n_{\alpha}^\dagger p_{\alpha} \]

- Expect
  \[ [\hat{H}_S, \hat{T}^\pm] = 0 \]

- Consider
  \[ \hat{T}_3 = \frac{1}{2} [\hat{T}^+, \hat{T}^-] = \frac{1}{2} \sum_{\alpha \beta} (p_{\alpha}^\dagger n_{\alpha} n_{\beta}^\dagger p_{\beta} - n_{\beta}^\dagger p_{\beta} p_{\alpha}^\dagger n_{\alpha}) \]
  \[ = \frac{1}{2} \sum_{\alpha \beta} (p_{\alpha}^\dagger p_{\beta} \delta_{\alpha, \beta} - n_{\beta}^\dagger n_{\alpha} \delta_{\alpha, \beta}) = \frac{1}{2} \sum_{\alpha} (p_{\alpha}^\dagger p_{\alpha} - n_{\alpha}^\dagger n_{\alpha}) \]

- will also commute with \( H_S \)
Isospin

• Check \( [\hat{T}_3, \hat{T}^\pm] = \pm \hat{T}^\pm \)

• Then operators

\[
\begin{align*}
\hat{T}_1 &= \frac{1}{2} (\hat{T}^+ + \hat{T}^-) \\
\hat{T}_2 &= \frac{1}{2i} (\hat{T}^+ - \hat{T}^-) \\
\hat{T}_3
\end{align*}
\]

obey the same algebra as \( J_x, J_y, J_z \) so spectrum identical and \( \hat{H}_S, \hat{T}^2, \hat{T}_3 \) simultaneously diagonal!

proton \( |r m_s\rangle_p = |r m_s m_t = \frac{1}{2}\rangle \)

neutron \( |r m_s\rangle_n = |r m_s m_t = -\frac{1}{2}\rangle \)

For this doublet \( T^2 |r m_s m_t\rangle = \frac{1}{2} \left( \frac{1}{2} + 1 \right) |r m_s m_t\rangle \)

and \( T_3 |r m_s m_t\rangle = m_t |r m_s m_t\rangle \)

States with total isospin constructed as for angular momentum
Two-particle states and interactions

- Pauli principle has important effect on possible states
- Free particles $\Rightarrow$ plane waves
- Eigenstates of $T = \frac{p^2}{2m}$ notation (isospin)
- Use box normalization (should be familiar)
- Nucleons $|p\ s = \frac{1}{2}\ m_s\ t = \frac{1}{2}\ m_t\rangle \equiv |pm_sm_t\rangle$
- Electrons, $^3$He atoms $|p\ s = \frac{1}{2}\ m_s\rangle \equiv |pm_s\rangle$
- Bosons with zero spin ($^4$He atoms) $|p\rangle$

- Use successive basis transformations for two-nucleon states to survey angular momentum restrictions
- Total spin & isospin; CM and relative momentum; orbital angular momentum relative motion; total angular momentum
Antisymmetric two-nucleon states

- Start with

\[ |p_{1m_{s1}m_{t1}}; p_{2m_{s2}m_{t2}}\rangle = \frac{1}{\sqrt{2}} \left\{ |p_{1m_{s1}m_{t1}}\rangle |p_{2m_{s2}m_{t2}}\rangle - |p_{2m_{s2}m_{t2}}\rangle |p_{1m_{s1}m_{t1}}\rangle \right\} \]

\[ = \frac{1}{\sqrt{2}} \sum_{SM_S} \sum_{TM_T} \left\{ (\frac{1}{2} m_{s1} \frac{1}{2} m_{s2} |SM_S\rangle (\frac{1}{2} m_{t1} \frac{1}{2} m_{t2} |TM_T\rangle |p_1 p_2 S M_S T M_T\rangle - (\frac{1}{2} m_{s2} \frac{1}{2} m_{s1} |SM_S\rangle (\frac{1}{2} m_{t2} \frac{1}{2} m_{t1} |TM_T\rangle |p_2 p_1 S M_S T M_T\rangle \right\} \]

- then

\[ P = p_1 + p_2 \]
\[ p = \frac{1}{2} (p_1 - p_2) \]

- and use

\[ |p\rangle = \sum_{LM_L} |pLM_L\rangle \langle LM_L|\hat{p}\rangle = \sum_{LM_L} |pLM_L\rangle Y^*_{LM_L}(\hat{p}) \]
\[ |-p\rangle = \sum_{LM_L} |pLM_L\rangle \langle LM_L|-\hat{p}\rangle = \sum_{LM_L} |pLM_L\rangle (-1)^L Y^*_{LM_L}(\hat{p}) \]

\[ Y^*_{LM_L}(-\hat{p}) = Y^*_{LM_L}(\pi - \theta_p, \phi_p + \pi) = (-1)^L Y^*_{LM_L}(\hat{p}) \]

- as well as

\[ (\frac{1}{2} m_{s2} \frac{1}{2} m_{s1} |SM_S\rangle = (-1)^{\frac{1}{2} + \frac{1}{2} - S} (\frac{1}{2} m_{s1} \frac{1}{2} m_{s2} |SM_S\rangle \]
\[ (\frac{1}{2} m_{t2} \frac{1}{2} m_{t1} |TM_T\rangle = (-1)^{\frac{1}{2} + \frac{1}{2} - T} (\frac{1}{2} m_{t1} \frac{1}{2} m_{t2} |TM_T\rangle \]

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Antisymmetry constraints for two nucleons

• Summarize

\[
|p_1 m_{s_1} m_{t_1}; p_2 m_{s_2} m_{t_2}\rangle =
\frac{1}{\sqrt{2}} \sum_{S M S T M T L M L} \left( \frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} |S M S\rangle \left( \frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} |T M T\rangle Y^*_{L M L}(\hat{p})
\times \left[ 1 - (-1)^{L+S+T} \right] |P p L M L S M S T M T\rangle
\right)
\]

\[
= \frac{1}{\sqrt{2}} \sum_{S M S T M T L M L J M J} \left( \frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} |S M S\rangle \left( \frac{1}{2} m_{t_1} \frac{1}{2} m_{t_2} |T M T\rangle Y^*_{L M L}(\hat{p})
\times (L M L S M S |J M J\rangle \left[ 1 - (-1)^{L+S+T} \right] |P p (L S) J M J T M T\rangle
\right)
\]

• \(L + S + T\) must be odd!

  - Notation

\[
\begin{align*}
T=0 & & T=1 \\
3S_1 - 3D_1 & & 1S_0 \\
1P_1 & & 3P_0 \\
3D_2 & & 3P_1 \\
& & 3P_2 - 3F_2 \\
& & 1D_2 \\
& & ...
\end{align*}
\]
Two electrons and two spinless bosons

• Remove isospin

\[
|p_1 m_{s_1}; p_2 m_{s_2}\rangle = \frac{1}{\sqrt{2}} \sum_{S M S L M L} \left( \frac{1}{2} m_{s_1} \frac{1}{2} m_{s_2} \right) |S M_S\rangle Y^*_{L M L}(\hat{p}) \times [1 + (-1)^{L+S}] \left| P p L M L S M_S \right\rangle
\]

• \( L + S \) even!

• Two spinless bosons

\[
|p_1; p_2\rangle = \frac{1}{\sqrt{2}} \sum_{L M L} Y^*_{L M L}(\hat{p}) [1 + (-1)^L] \left| P p L M L \right\rangle
\]

• \( L \) even!
Nuclei

• Different shells only Clebsch-Gordan constraint

• Uncoupled states in the same shell \[ |\Phi_{jm,jm'}\rangle = a_{jm}^\dagger a_{jm'}^\dagger |\Phi_0\rangle \]

• Coupling
\[ |\Phi_{jj,JM}\rangle = \sum_{mm'} (j \ m \ j' \ m' | J \ M) |\Phi_{jm,jm'}\rangle = \sum_{mm'} (j \ m' \ j \ m | J \ M) |\Phi_{jm',jm}\rangle \]
\[ = \sum_{mm'} (-1)^{2j-J} (j \ m \ j' \ m' | J \ M) (-1) |\Phi_{jm,jm'}\rangle \]
\[ = (-1)^J \sum_{mm'} (j \ m \ j' \ m' | J \ M) |\Phi_{jm,jm'}\rangle \]
\[ = (-1)^J |\Phi_{jj,JM}\rangle \]

• Only even total angular momentum

• With isospin
\[ |\Phi_{jj,JM,TM_T}\rangle = \sum_{mm'm_t'm_t'} (j \ m \ j' \ m' | J \ M) (\frac{1}{2} \ m_t \ \frac{1}{2} \ m_t' | T \ M_T) |\Phi_{jm,mm_t',jm'm_t}\rangle \]
\[ = (-1)^{J+T+1} |\Phi_{jj,JM,TM_T}\rangle \]

• J+T odd!
$^{40}\text{Ca} + \text{two nucleons}$

- Spectrum

Excitation Energy [MeV]

- $^{42}\text{Ca}$: $0^+$
- $^{42}\text{Sc}$: $0^+$
- $^{42}\text{Ti}$: $0^+$

Levels:
- $0^+$
- $2^+$
- $4^+$
- $6^+$
- $(2^+)$
- $(7^+)$
- $(5^+)$
- $(3^+)$
Two fermions outside closed shells

- Atoms: two different orbits ⇒ usual Clebsch-Gordan coupling and no other restrictions
- Two electrons in the same orbit
  \[ |\Phi_{\ell m_\ell m_s,\ell m'_\ell m'_s}\rangle = a_{\ell m_\ell m_s}^\dagger a_{\ell m'_\ell m'_s}^\dagger |\Phi_0\rangle \]
- Coupling
  \[ |\Phi_{L M L, S M S}\rangle = \sum_{m_\ell m'_\ell m_s m'_s} (\ell m_\ell \ell m'_\ell |L M_L\rangle (\frac{1}{2} m_s \frac{1}{2} m'_s |S M_S\rangle |\Phi_{\ell m_\ell m_s,\ell m'_\ell m'_s}\rangle \]
  \[ = (-1)^{L+S} |\Phi_{\ell \ell, L M L, S M S}\rangle \]
- \(L+S\) even
- **Carbon** \((2p)^2\)
  - L=0 ⇒ S=0
    \[ \text{# of states} = 1 \]
    \[ \text{notation} = 1S \]
  - L=1 ⇒ S=1
    \[ \text{# of states} = 9 \]
    \[ \text{notation} = 3P \text{ g.s.} \]
  - L=2 ⇒ S=0
    \[ \text{# of states} = 5 \]
    \[ \text{notation} = 1D \]
- Why $S=1$ g.s.?
- D below S?