Optional problem set 1

Laplacian in 3-dimensions in Cartesian coordinate:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
\]  

(1)

The Cartesian co-ordinates \((x, y, z)\) are related to spherical polar co-ordinates \((r, \theta, \phi)\) by

\[
x = r \cos \phi \sin \theta \\
y = r \sin \phi \sin \theta \\
z = r \cos \theta.
\]

To translate (1) into a differential equation involving \((r, \theta, \phi)\) we need the following partial derivatives:

\[
\frac{\partial x}{\partial r} = \cos \phi \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \phi \sin \theta, \quad \frac{\partial z}{\partial r} = \cos \theta, \\
\frac{\partial x}{\partial \theta} = r \cos \phi \cos \theta, \quad \frac{\partial y}{\partial \theta} = r \sin \phi \cos \theta, \quad \frac{\partial z}{\partial \theta} = -r \sin \theta, \\
\frac{\partial x}{\partial \phi} = -r \sin \phi \sin \theta, \quad \frac{\partial y}{\partial \phi} = r \cos \phi \sin \theta, \quad \frac{\partial z}{\partial \phi} = 0.
\]

Using these the chain rule for differentiation implies that

\[
\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} = \cos \phi \sin \theta \frac{\partial}{\partial x} + \sin \phi \sin \theta \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} = r \cos \phi \cos \theta \frac{\partial}{\partial x} + r \sin \phi \cos \theta \frac{\partial}{\partial y} - r \sin \theta \frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial}{\partial z} = -r \sin \phi \sin \theta \frac{\partial}{\partial x} + r \cos \phi \sin \theta \frac{\partial}{\partial y}.
\]

These can be inverted, by taking linear combination with trigonometric function for example, to express partial derivatives of Cartesian co-ordinates in terms of polar co-ordinates:

\[
\frac{\partial}{\partial x} = \cos \phi \sin \theta \frac{\partial}{\partial r} + \cos \phi \cos \theta \frac{\partial}{\partial \theta} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial y} = \sin \phi \sin \theta \frac{\partial}{\partial r} + \sin \phi \cos \theta \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r} \frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \phi}.
\]  

(2)

One way of deriving the Laplacian in 3-dimensional spherical polars is to expand the unit vectors \(\hat{r}, \hat{\theta} \text{ and } \hat{\phi}\) in the directions of increasing \(r, \theta\) and \(\phi\) respectively, in terms of the Cartesian unit vectors \(\hat{x}, \hat{y} \text{ and } \hat{z}\):

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}, \\
\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}, \\
\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}.
\]  

(3)

Credit: http://www.thphys.nuim.ie/Notes/MP469/
These can be inverted to give
\[
\begin{align*}
\hat{x} &= \sin \theta \cos \phi \, \hat{r} + \cos \theta \cos \phi \, \hat{\theta} - \sin \phi \, \hat{\phi}, \\
\hat{y} &= \sin \theta \sin \phi \, \hat{r} + \cos \theta \sin \phi \, \hat{\theta} + \cos \phi \, \hat{\phi}, \\
\hat{z} &= \cos \theta \, \hat{r} - \sin \theta \, \hat{\theta}.
\end{align*}
\] (4)

Then define the vector differential operator
\[
\nabla := \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}
\] (5)

and the Laplacian can be written as $\nabla^2 = \nabla \cdot \nabla$, using the vector dot product. Putting (2) and (4) in (5), and using standard trigonometric identities, gives
\[
\nabla := \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.
\] (6)

Now to calculate $\nabla^2 = \nabla \cdot \nabla$ in spherical polars we must be careful since the polar unit vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are not constant. From (3)
\[
\begin{align*}
\frac{\partial}{\partial r} \hat{r} &= 0, & \frac{\partial}{\partial r} \hat{\theta} &= 0, & \frac{\partial}{\partial r} \hat{\phi} &= 0, \\
\frac{\partial}{\partial \theta} \hat{r} &= \hat{\theta}, & \frac{\partial}{\partial \theta} \hat{\theta} &= -\hat{r}, & \frac{\partial}{\partial \theta} \hat{\phi} &= 0, \\
\frac{\partial}{\partial \phi} \hat{r} &= -\sin \theta \, \hat{\phi}, & \frac{\partial}{\partial \phi} \hat{\theta} &= \cos \theta \, \hat{\phi}, & \frac{\partial}{\partial \phi} \hat{\phi} &= -\sin \theta \, \hat{r} - \cos \theta \, \hat{\theta}.
\end{align*}
\]

Using these in (6) the Laplace differential operator in equation (1) can be expressed directly in terms of spherical polar co-ordinates:
\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
= \frac{1}{r} \frac{\partial^2 (ru)}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}.
\]