Problem set 6

1.

\[ a = 100; \]
\[ m = 0.1; \]
\[ \hbar = 1; \]

\[ \psi_1[x_] := \sqrt{\frac{2}{a}} \cos(\pi x / a); \]

\[ \psi_2[x_] := \sqrt{\frac{2}{a}} \sin(2\pi x / a); \]

\[ \text{Plot}[\{\psi_1[x], \psi_2[x]\}, \{x, -50, 50\}, \text{AxesLabel} \to \{"x"\}, \text{PlotLegends} \to \{"\psi_1", "\psi_2"\}] \]

\[ \int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_1^* x \psi_1 \, dx \quad (*) \quad <\psi_1 | \hat{x} | \psi_1> (*) \]

\[ \int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_2^* x \psi_2 \, dx \quad (*) \quad <\psi_2 | \hat{x} | \psi_2> (*) \]

\[ (*) \quad \text{Alternatively, use symmetry arguments: since } \psi^* \psi \text{ is an even function, and } x \text{ is an odd function, the integration of } (\psi^* x \psi), \]

\[ \text{which is an odd function, over the range } (-a/2, a/2), \text{ is 0.} \quad (*) \]

2. (a)
In[9]:= \text{wavefn}$t0$[x_] := \frac{1}{\sqrt{2}} (\psi_1[x] + \psi_2[x]);

Plot[wavefn$t0[x], \{x, -50, 50\}, \text{AxesLabel} \to \{"x"\},
\text{PlotLegends} \to \{"\psi"\}] (* Plot of } \psi(x,0) \) *)
Plot[Conjugate[wavefn$t0[x]] wavefn$t0[x],
\{x, -50, 50\}, \text{AxesLabel} \to \{"x", "|\psi(x,0)|^2"\}]
(* Plot of the probability density } |\psi(x,0)|^2 \) *)
(* Since the probability density is higher around the region (-10, 40),
it is most likely to find the particle in that region at time } t=0. \)

Out[10]=

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Probability density of } \psi(x,0) \caption{(b)}
\end{figure}

In[12]= \int_{-\frac{a^2}{2}}^{\frac{a^2}{2}} \text{Conjugate[wavefn$t0[x]] wavefn$t0[x]} \ dx

(* Check normalization of } \psi(x,0) \) *)
N[\int_{-\frac{a^2}{2}}^{\frac{a^2}{2}} \text{Conjugate[wavefn$t0[x]] x wavefn$t0[x]} \ dx]

(* Expectation value of position. The value is around where
the particle is most likely to find, which is reasonable. *)

Out[12]= 1

3. (a) The time dependence of the eigenfunction is \(e^{-iE_n t/\hbar}\). Therefore, the angular frequency of the eigenfunction is \(\omega = \frac{E_n}{\hbar} = \frac{\hbar n^2}{2ma^2}\). Hence, the angular frequency \(\omega_1 = \frac{\hbar n^2}{2ma^2}\), and for the second eigenfunction, the angular frequency \(\omega_2 = \frac{2\pi \hbar}{ma^2}\).

The wavefunctions evolve in time:

\[
\psi_1(x, t) = \psi_1(x) e^{-i\omega_1 t} = \sqrt{\frac{2}{a}} \cos(\pi x/a) e^{-i\frac{a^2 \pi}{ma^2} t};
\]

\[
\psi_2(x, t) = \psi_2(x) e^{-i\omega_2 t} = \sqrt{\frac{2}{a}} \sin(2\pi x/a) e^{-i\frac{2a^2 \pi}{ma^2} t}.
\]

(b)

\[
\text{wavefn}[x_, t_] := \frac{1}{\sqrt{2}} \left( \psi_1[x] e^{-i\frac{a^2 \pi}{ma^2} t} + \psi_2[x] e^{-i\frac{2a^2 \pi}{ma^2} t} \right);
\]

\[
\text{wavefn}[x, 0] = \text{wavefn}[x, 0] (* \text{Neglecting the 0.i term, this gives the same wavefunction as wavefn}[x, 0] *. *)
\]

\[
(0.141421 + 0.i) \cos\left[\frac{\pi x}{100}\right] + (0.141421 + 0.i) \sin\left[\frac{\pi x}{50}\right]
\]

\[
\sqrt{2}
\]

\[
\frac{\cos\left[\frac{\pi x}{100}\right] + \sin\left[\frac{\pi x}{50}\right]}{5\sqrt{2}} + \frac{\sin\left[\frac{\pi x}{100}\right]}{5\sqrt{2}}
\]

\[
\sqrt{2}
\]

(c)

\[
\text{DynamicModule}[\{t\}, \{\text{Slider}[	ext{Dynamic}[t], \{0, 600, 1\}]\},
\]

\[
"t = " \text{Dynamic}[t], \text{Dynamic}[\text{Plot}[\text{Conjugate}[\text{wavefn}[x, t]] \text{ wavefn}[x, t],
\]

\[
\{x, -50, 50\}, \text{AxesLabel} \to \{"x", "|\psi(x, t)|^2\}\}]
\]

As time goes on, the most probable position of the particle moves from around 20 to -20 gradually. Its period is roughly \(T = 420\). The particle roughly moves around \((-20, 20)\), hence its amplitude of oscillations is about 20.

(d)
\[ \text{expected}\{x, t\} := \int_{-\frac{a}{2}}^{\frac{a}{2}} \text{Conjugate}[\text{wavefn}[x, t]] \times \text{wavefn}[x, t] \, dx; \]

\[ \text{expected}\{x, 0\} \]

(* which is consistent with the last part of question 2 *)

\[ \text{Plot}[\text{expected}\{x, t\}, \{t, 0, 600\}, \text{AxesLabel} \rightarrow \{"t", "<\dot{x}>"}] \]

The particle moves back and forth about the origin. Its amplitude is around 18, and its period of oscillation is around \((530 - 105 =)\) 425.

4.
\[
\langle \psi | \hat{x} | \psi \rangle = \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{\sqrt{2}} \left( \psi_1 e^{-\frac{i\pi}{2a} x} t + \psi_2 e^{-\frac{i2\pi}{a} x} t \right) \times \frac{1}{\sqrt{2}} \left( \psi_1 e^{-\frac{i\pi}{2a} x} t + \psi_2 e^{-\frac{i2\pi}{a} x} t \right) dx
\]

\[
= \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left( \psi_1^* \psi_1 + \psi_1^* \psi_2 + \psi_2^* \psi_1 + \psi_2^* \psi_2 \right) e^{i \left( -\frac{\pi}{2a} x^2 - \frac{2\pi}{a} x t \right)} dx
\]

(* We are already familiar with the result: \( \int_{-\frac{a}{2}}^{\frac{a}{2}} \psi_n^* x \psi_n dx = 0. * \)

\[
= \frac{1}{2} e^{i \left( -\frac{\pi}{2a} x^2 - \frac{2\pi}{a} x t \right)} t \int_{-\frac{a}{2}}^{\frac{a}{2}} (\psi_1^* x \psi_2 + \psi_2^* x \psi_1) dx
\]

\[
= \frac{1}{2} e^{i \left( -\frac{\pi}{2a} x^2 - \frac{2\pi}{a} x t \right)} t \int_{-\frac{a}{2}}^{\frac{a}{2}} \left( \cos \left( \frac{\pi}{a} x \right) x \sin \left( \frac{2\pi}{a} x \right) + \sin \left( \frac{2\pi}{a} x \right) x \cos \left( \frac{\pi}{a} x \right) \right) dx
\]

(* Change variable \( \frac{2a}{\pi} x = \theta \) * )

\[
= \frac{1}{a} e^{i \left( -\frac{\pi}{2a} x^2 - \frac{2\pi}{a} x t \right)} t \int_{-\frac{a}{2}}^{\frac{a}{2}} \left( \cos \theta \frac{a}{\pi} \theta \sin (2\theta) + \sin (2\theta) \frac{a}{\pi} \theta \cos (\theta) \right) \frac{a}{\pi} d\theta
\]

\[
= \frac{1}{a} a^2 e^{i \left( -\frac{\pi}{2a} x^2 - \frac{2\pi}{a} x t \right)} t \int_{-\frac{a}{2}}^{\frac{a}{2}} \left( \cos \theta \frac{a}{\pi} \theta \sin (2\theta) + \sin (2\theta) \frac{a}{\pi} \theta \cos (\theta) \right) \frac{a}{\pi} d\theta
\]

\[
= \frac{1}{9\pi^2} e^{i \frac{4a \pi^2}{9}} t
\]

(* The oscillation period is \( \tau = \frac{4 m a^2}{3 \pi \hbar} \), and the amplitude of the expected position is \( \frac{16 a^9}{9 \pi^2} \). *)

\[
= \frac{16 \times 100}{9\pi^2} e^{i \frac{4 \times 0.1 \times 10^2}{9} t}
\]

\[
= \frac{18.0127 e^{-0.0148 i t}}{}
\]

(* The amplitude of the oscillation of position expectation value is indeed 18.0127, and the period is \( \tau = \frac{2 \pi}{0.0148} = 424.4 \), which is about the same as our estimation. *)