

# Axiomatic Quantum Field Theory in Curved Spacetime

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## Wightman Axioms in Minkowski Spacetime

- The states of the theory are unit rays in a Hilbert space,  $\mathcal{H}$ , that carries a unitary representation of the Poincare group.
- The 4-momentum (defined by the action of the Poincare group on the Hilbert space) is positive, i.e., its spectrum is contained within the closed future light cone (“spectrum condition”).
- There exists a unique, Poincare invariant state (“the vacuum”).
- The quantum fields are operator-valued distributions defined on a dense domain  $\mathcal{D} \subset \mathcal{H}$  that is both

Poincare invariant and invariant under the action of the fields and their adjoints.

- The fields transform in a covariant manner under the action of Poincare transformations.
- At spacelike separations, quantum fields either commute or anticommute.

## Difficulties with Extending the Wightman Axioms to Curved Spacetime

- A generic curved spacetime will not possess any symmetries at all, so one certainly cannot require “Poincare invariance/covariance” or invariance under any other type of spacetime symmetry.
- There exist unitarily inequivalent Hilbert space constructions of free quantum fields in spacetimes with a noncompact Cauchy surface and (in the absence of symmetries of the spacetime) none appears “preferred”.
- In a generic curved spacetime, there is no “preferred” choice of a vacuum state.

- There is no analog of the spectrum condition in curved spacetime that can be formulated in terms of the “total energy-momentum” of the quantum field.

Thus, of all of the Wightman axioms, only the last one (commutativity or anticommutativity at spacelike separations) generalizes straightforwardly to curved spacetime.

## Overcoming These Difficulties

- The difficulties that arise from the existence of unitarily inequivalent Hilbert space constructions of quantum field theory in curved spacetime can be overcome by formulating the theory via the algebraic framework.
- The difficulties that arise from the lack of a useful notion of the total energy-momentum of the quantum field can be overcome by reformulating the spectrum condition in terms of the local in spacetime “positive frequency” properties of the singularity structure of the correlation functions and time-ordered products of quantum fields (“microlocal spectrum condition”).

- Many aspects of the requirement of Poincare invariance of the quantum fields can be replaced by the requirement that the quantum fields be locally and covariantly constructed out of the metric.

## Local and Covariant Fields

We wish to impose the requirement that a quantum field  $\phi$  at point  $x$  “be locally and covariantly constructed out of the spacetime geometry” in an arbitrarily small neighborhood of  $x$ . More precisely, if we have an isometric embedding  $\chi : \mathcal{O} \rightarrow \mathcal{O}'$  of a region,  $\mathcal{O}$ , of the spacetime  $(M, g_{ab})$  into a region  $\mathcal{O}'$ , of the spacetime  $(M', g'_{ab})$ , we require that this embedding induce an isomorphism of the local field subalgebras associated with  $\mathcal{O}$  and  $\mathcal{O}'$ . We further demand that under this isomorphism, the field  $\phi(f)$  in  $\mathcal{O}$  be taken into the field  $\phi(\chi^* f)$  in  $\mathcal{O}'$ .

We can isometrically embed all of Minkowski spacetime

into itself by a Poincare transformation. The above condition then requires the field definitions in Minkowski spacetime to be Poincare invariant. **The above condition contains much of the essential content of Poincare invariance, but is applicable to arbitrary curved spacetimes.**

## Generalizing the Wightman Axioms to Curved Spacetime

- The states of the theory are unit rays in a Hilbert space,  $\mathcal{H}$ , that carries a unitary representation of the Poincare group.
- The 4-momentum (defined by the action of the Poincare group on the Hilbert space) is positive, i.e., its spectrum is contained within the closed future light cone (“spectrum condition”).
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Poincare invariant and invariant under the action of the fields and their adjoints.

- The fields transform in a covariant manner under the action of Poincare transformations.
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What is the appropriate replacement in curved spacetime of the requirement that there exist a Poincare invariant state in Minkowski spacetime?

## The Operator Product Expansion

An *operator product expansion* is a short-distance asymptotic formula for products of fields near point  $x$  in terms of fields defined at  $x$ . For example, for a free Klein-Gordon field in curved spacetime, we have

$$\phi(x)\phi(y) = H(x, y)\mathbf{1} + \phi^2(x) + \dots$$

where  $H$  is a locally and covariantly constructed Hadamard distribution and “...” has higher scaling degree than the other terms (i.e., it goes to zero more rapidly in the coincidence limit). Hollands has proven that an operator product expansion exists for free fields in curved spacetime and holds order-by-order in

perturbation theory for interacting fields in curved spacetime.

We believe that the existence of an operator product expansion satisfying certain properties should be elevated to the status of a fundamental property of quantum fields. **The requirement that such an operator product expansion exist provides a suitable replacement for the requirement of existence of a Poincare invariant state!** In particular, with this assumption, we can prove a spin-statistics theorem and a CPT theorem in curved spacetime.

## Conclusions and Outlook

We believe that we have provided a suitable precise notion of an interacting quantum field in curved spacetime. This notion captures much of the essential content of the Wightman axioms, but reformulates the ideas in a very different way. In particular, we have completely dispensed with the requirement of the existence of a preferred vacuum state.

The well known difficulties with the perturbative construction of interacting quantum field theory in curved spacetime appear to be associated with the non-analytic dependence of the vacuum state on the coupling parameters. **It would be interesting to**

re-examine the convergence of perturbation theory within  
our new framework.