Perturbation Theory for Fat-link Fermion Actions

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We discuss weak coupling perturbation theory for lattice actions in which the fermions couple to smeared gauge links. The normally large integrals that appear in lattice perturbation theory are drastically reduced. Even without detailed calculation, it is easy to determine to good accuracy the scale of the logarithms that appear in cases where an anomalous dimension is present. We describe several 1-loop examples for fat-link Wilson and clover fermions. including the additive mass shift, the relation between the lattice and $\overline{MS}$ quark masses, and the axial current renormalization factor (Z_A) for light-light and static-light currents.

Smearing or "fattening" the gauge links in a fermion action [1] has several advantages. Indeed, with sufficient fattening:

- Exceptional configurations are suppressed.
- Additive mass renormalization is small.
- Finite matching constants (Z factors) are very close to 1.
- We expect that tree level $\mathcal{O}(a)$ improvement (clover coefficient $C_{SW} = 1$) is close to all-orders (in g) $\mathcal{O}(a)$ improvement.

The last three features are closely related and can be understood in perturbation theory. Furthermore, these actions are already being used in Monte-Carlo computations (*e.g.*, by MILC [2]), so it is important, as a first step, to have perturbative evaluations of the renormalization constants. Non-perturbative evaluations, where possible, are of course a very important next step.

In the present work, we consider the ordinary Wilson and clover fermion actions in which the gauge links are fattened by APE smearing [3]. The link after $m+1$ smearings is related to the link after m smearings by

$$V_\mu^{(m+1)}(x) = \mathcal{P}\Big((1-\alpha)V_\mu^{(m)}(x) + \frac{\alpha}{6}\sum_{\nu\neq\mu}\big[V_\nu^{(m)}(x)V_\mu^{(m)}(x+\hat\nu)V_\nu^{\dagger(m)}(x+\hat\mu) + V_\nu^{\dagger(m)}(x-\hat\nu)V_\mu^{(m)}(x-\hat\nu)V_\nu^{(m)}(x-\hat\nu+\hat\mu)\big]\Big)$$

where $\mathcal{P}$ is the projection back into SU(3) and $V_\mu^0(x) = U_\mu(x)$, the original ("thin") link.

If one fixes α and performs a fixed number of smearings, n, as $a \to 0$, the action remains local and the continuum limit will be unaffected. In practice, MILC has been trying $\alpha = 0.45$ and $n = 10$ for heavy-light decay constants and form factors; we use these values for all "fat" results quoted in this paper. (Unless specified otherwise, we take $C_{SW} = 1$ in clover computations.)

We are interested in computing 2- and 4-quark operator renormalization/matching constants at one loop. For such computations, only the linear part of the relation between fat and thin links is relevant ($V_\mu^{(m)}(x) \equiv \exp(iaA_\mu^{(m)}(x))$:

$$A_\mu^{(1)}(x) = \sum_{y,\nu} h_{\mu\nu}(y) A_\nu^{(0)}(x+y) \ . \qquad (1)$$

Quadratic terms in (1), which would be relevant for tadpole graphs only, appear as commutators and therefore do not contribute, since tadpoles are symmetric in the two gluons.

In momentum space, eq. (1) becomes

$$A_\mu^{(1)}(q) = \sum_\nu \tilde h_{\mu\nu}(q) A_\nu^{(0)}(q) \ , \qquad (2)$$

where

$$\tilde h_{\mu\nu}(q) = f(q)(\delta_{\mu\nu} - \frac{\hat q_\mu \hat q_\nu}{\hat q^2}) + \frac{\hat q_\mu \hat q_\nu}{\hat q^2} \ , \qquad (3)$$

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with $\hat{q}_\mu = \frac{2}{a}\sin(\frac{aq_\mu}{2})$ and $f(q) = 1 - \frac{\alpha}{6}\hat{q}^2$.

After n smearings, $\tilde{h}_{\mu\nu}(q)$ becomes $\tilde{h}^n_{\mu\nu}(q)$, which is just $\tilde{h}_{\mu\nu}$ with f replaced by f^n. If $0 \leq \alpha \leq 0.75$, then $|f| \leq 1$ over the entire Brillouin zone, and the factor f^n will, for large n, strongly suppress values of q outside of a small ball around $q = 0$.

For small α, it is easy to find the effective range of the fattening:

$$f^n(q) \sim \exp(-\frac{n\alpha}{6}\hat{q}^2) \Rightarrow \langle x^2 \rangle \sim \frac{n\alpha}{3}a^2 \qquad (4)$$

Thus, even an $n = 10$, $\alpha = 0.45$ fat link has a mean size of only a couple of lattice spacings.

The longitudinal part of A_μ in eq. (2) is unaffected by smearing, because it is a pure gauge, for which parallel transport is independent of path.

The Feynman rules are easily derived from eqs. (2,3). Each quark-gluon vertex gets a form factor $\tilde{h}^n_{\mu\nu}(q)$, where q is the gluon momentum. If all gluon lines start and end on fermion lines, then, effectively, the gluon propagator changes by $D_{\mu\nu} \longrightarrow \tilde{h}^n_{\mu\lambda} D_{\lambda\sigma} \tilde{h}^n_{\sigma\nu}$. The form of $\tilde{h}_{\mu\nu}$ shows that Landau gauge is the most natural gauge for fat-link perturbation theory, since the Landau propagator will kill the longitudinal part of $\tilde{h}$.

In many cases it is now easy to convert ordinary thin-link perturbation theory to the fat-link case. In particular, consider some pure lattice quantity (*e.g.*, the additive mass shift, or Z_A for two light quarks) computed for thin links at one loop. The answer is of the form $\int \frac{d^4q}{(2\pi)^4} I_{latt}(q)$. If the thin-link calculation was done in Landau gauge with the loop momentum along the gluon line, then the fat-link result is obtained simply by replacing $I_{latt}(q) \to I_{latt}(q) f^{2n}(q)$.

However, if the thin-link calculation was done in Feynman gauge with the loop momentum along the gluon line, the fat-link Landau gauge computation will have some new terms, coming from the $f^n \frac{\hat{q}_\mu \hat{q}_\nu}{\hat{q}^2}$ part of $\tilde{h}^n_{\mu\nu}(q)$. When $f = 1$, the sum of such terms must vanish by gauge invariance. If the cancellation occurs at the level of the integrands, multiplying by f^{2n} will not affect it, and the change from thin to fat links is again accomplished by $I_{latt}(q) \to I_{latt}(q) f^{2n}(q)$. In the cases we have checked (additive mass shift, multiplicative mass renormalization), the cancellation does indeed occur at the level of the integrands. However, we know of no general proof that this must be the case. If the cancellation required integration by parts, for example, the fat-link computation would contain some new terms not present in the thin-link case.

We can now perform the lattice integrals numerically. An important first example is the simple gluon tadpole ($I_{latt} = \frac{1}{\hat{q}^2}$). We find that it has the value 0.1547 in the thin-link case, but only 0.0044 for fat links. The form factor f^{2n} has completely suppressed the UV part of the Brillouin zone, which gives the dominant contribution to the tadpole. Thus tadpole improvement is unnecessary for fat links, at least with this much smearing.

Another example is $Z_A^{\rm ll}$, the axial current renormalization constant for two massless quarks. Writing $Z_A^{\rm ll} = 1 + \frac{g^2 C_F}{16\pi^2}\zeta_A^{\rm ll}$, we have

$$\zeta_A^{\rm ll} = \begin{cases} -15.80, & \text{thin Wilson} \\ -13.80, & \text{thin clover} \\ -0.24, & \text{fat clover} \end{cases} \qquad (5)$$

(Here clover currents are local: improved by the factor $1 + m_0 = 1 + \frac{1}{2\kappa} - \frac{1}{2\kappa_c}$, not by derivative terms.) Again, the form factor has suppressed most of the integration region, making the integral small. With this much fattening, all such pure lattice integrals should be highly suppressed. This explains why the additive mass renormalization is so small, and why a clover coefficient $C_{SW} = 1$ should be very close to the nonperturbatively improved value — certainly the tadpole-improved C_{SW} is extremely close to 1.

The matching of a divergent operator (*i.e.*, one with an anomalous dimension) is slightly more complicated. In this case, a standard way to express the lattice result in the thin case is:

$$\int \frac{d^4q}{(2\pi)^4} \left[I_{latt}(q) - I_{cont}(q, 0) \right] + \int \frac{d^4q}{(2\pi)^4} I_{cont}(q, a\lambda) \; , \qquad (6)$$

where λ stands for some IR cutoff, such as a quark mass or momentum, or a gluon mass inserted by hand. $I_{cont}(q, a\lambda)$ is a simple continuum-like integrand which has the same IR behavior as the lattice integrand, but which can be integrated analytically, giving the explicit $\ln(a\lambda)$ behavior.

In the fat case, the analogous formula is

$$\begin{aligned}\int \tfrac{d^4q}{(2\pi)^4} f^{2n} \left[I_{latt}\left(q\right) - I_{cont}\left(q,0\right)\right] \\ + \int \tfrac{d^4q}{(2\pi)^4} I_{cont}(q, a\lambda) \\ + \int \tfrac{d^4q}{(2\pi)^4} I_{cont}\left(q,0\right)\left(f^{2n}-1\right) \ ,\end{aligned} \qquad (7)$$

where we have added and subtracted the integral of $f^{2n} I_{cont}$. This separates the result into three terms: (1) a complicated, regularized lattice integral, which is however suppressed for sufficient fattening and is thus numerically small, (2) the same continuum-like integral that appeared in the thin case, and (3) a simple integral involving the form factor, whose coefficient is determined by the anomalous dimension of the operator. Term (3) is not particularly small, since f^{2n} does not multiply the entire integrand.

An example is the matching of the lattice mass to the continuum $\overline{\mathrm{MS}}$ mass:

$$\mathcal{Z}_M(a\mu) = 1 + \frac{g^2 C_F}{16\pi^2}\left(\zeta_M - 6\log\left(a\mu\right)\right) \ . \qquad (8)$$

We get

$$\zeta_M = \begin{cases} 12.95, & \text{thin Wilson} \\ 19.31, & \text{thin clover} \\ -4.92, & \text{fat clover .} \end{cases} \qquad (9)$$

Z_A^{sl}, the axial current renormalization constant in the static-light case, also has an anomalous dimension. It does not separate as easily as eq. (7). However, the calculations, following [4], are straightforward. We find

$$Z_A^{\mathrm{sl}}(aM_B) = 1 + \frac{g^2 C_F}{16\pi^2}\left(\zeta_A^{\mathrm{sl}} + 3\log\left(aM_B\right)\right) \qquad (10)$$

$$\zeta_A^{\mathrm{sl}} = \begin{cases} -22.36, & \text{thin Wilson} \\ -16.41, & \text{thin clover} \\ 0.393, & \text{fat clover .} \end{cases} \qquad (11)$$

(The clover currents are local.)

The bad news about fat links is that, at least with the amount of fattening used in the examples above ($n = 10$, $\alpha = 0.45$), the lattice integrands are so suppressed at large q that they may become IR sensitive. A measure of the effective cutoff of the integrals is given by q^* [5]. With $C_{SW} = 1$, we find $q^* a = 0.71$ for Z_A^{ll}; while $q^* a = 1.05$ for Σ_0 (which gives the shift in κ_c).

More direct evidence that perturbation theory for Σ_0 is breaking down comes from a comparison with numerical results from the MILC collaboration. For $\beta = 5.6$, $m = .01$, $C_{SW} = 1.0$, $n = 10$, $\alpha = 0.45$, the numerical result is $\kappa_c = 0.1256(1)$; whereas 1-loop perturbation theory, with $q^* a = 1.05$, gives $\kappa_c = 0.1251$. Thus perturbation theory fails by a factor of six for $\kappa_c - 1/8$. There are however sensitive cancellations at 1-loop making Σ_0 anomalously small.

For the same data set, $Z_A^{\mathrm{ll}} = 0.99$ in perturbation theory. We guess this will be better behaved than Σ_0, because there are no delicate cancellations here. But even if $Z_A^{\mathrm{ll}} - 1$ is wrong by another factor of 6, the associated error in the final answer is only 5%, since Z^{ll} is so close to 1.

However, in cases with anomalous dimensions (*e.g.*, f_B), the perturbative corrections are not necessarily small, and a failure of perturbation theory could lead to large errors. Some non-perturbative calculations are needed to clarify this situation.

In retrospect, somewhat less fattening in the MILC running would probably have been preferable. For example, for the clover Z_A^{ll}, $n = 7$, $\alpha = 0.25$ gives $q^* a = 1.34$ instead of 0.71, and may still suppress exceptional configurations enough.

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