LATTICE CALCULATIONS OF DECAY CONSTANTS [a]

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Lattice attempts to compute the leptonic decay constants of heavy-light pseudoscalar mesons are described. I give a short historical overview of such attempts and then discuss some current calculations. I focus on three of the most important sources of systematic error: the extrapolation to the continuum, the chiral extrapolation in light quark mass, and the effects of quenching. I briefly discuss the "bag parameters" B_B and B_{B_s}, and then conclude with my expectations of the precision in decay constants and bag parameters that will be possible in the next few years.

1 Historical Overview

Over the past ten years, many groups have attempted to calculate f_B, f_{B_s}, f_D and f_{D_s} from lattice QCD. While the predictions for f_D and f_{D_s} have been quite stable, those for f_B and f_{B_s} have had a rather checkered history. It may be useful therefore to give a brief review of this history and to point out some of the problems that early computations encountered. I've divided my history into three periods: "ancient history," the "middle ages," and the "modern era." Let me caution at the outset that my divisions are somewhat arbitrary, and my placing of various papers into one period or another quite subjective. Furthermore, I do not pretend to have included all the papers of this subject; I have tried only to include what I consider "representative" works. For more comprehensive and detailed reviews, see Refs. [1,2].

1.1 Ancient History

While much that was useful was learned, the results for f_B and f_{B_s} that came from several early computations may now be ignored. First among these was a computation I was involved in: Ref. [3] ("BDHS"). We obtained a very low value of $f_B \sim 100$ MeV. The problem came from the treatment of heavy quarks on the lattice. While the results for f_D and f_{D_s} were reasonable, the extrapolation from the D to the B was strongly skewed by lattice artifacts, which arise whenever the heavy quark mass (m_Q) is comparable to the inverse lattice spacing (a^{-1}).

The static approximation on the lattice, in which $m_Q \to \infty$, is an approach that was introduced [4] to avoid the problems due to $m_Q a \sim 1$. However early

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computations within the static approximation [5,6,7] obtained very large answers: $f_B \sim 300$ MeV. I believe it is now generally accepted that the main problem here was contamination by excited states. The static approach is inherently very noisy, and without sophisticated techniques [8] that allow one to extract the signal at short times, one may easily be fooled by "false plateaus" into believing that excited state contamination is under control. In addition, it is now known that the result for f_B at the physical B meson is $\sim 20\%$ lower than at infinite b quark mass, and that the extrapolation to the continuum also tends to lower f_B.

1.2 Middle Ages

In this middle period I place papers which have no glaring problems, but which, either because of limited resources or limited focus, have in general rather crude systematic error estimates. This is not to say that systematic errors are ignored or treated cavalierly. Indeed, some of these errors are studied in great detail. However, even if one ignores the errors due to "quenching" (the neglect of virtual quark loops, *i.e.*, of sea quarks), none of these computations is able to include a complete study of all other significant sources of error. In Table 1, I list the papers that I have placed in this period, together with a list of relevant systematic issues.

A frequent concern about the papers in Table 1 is the lack of a reasonable way to estimate the errors coming from the extrapolation to the continuum. Often only one lattice spacing is available; when there are two, large statistical errors usually hide any systematic difference. Only two of the computations [9,10] consider a wide range of lattice spacings, which allows some control over the extrapolation. Note that, even for computations [11,12] which treat the heavy quark in the effective nonrelativistic QCD (NRQCD) framework, [13] an error estimate of discretization effects is needed because the light quark is still treated with a standard relativistic action. These discretization errors are not included in the truncation errors of the NQRCD action.

A second issue for several of the papers in Table 1 are the artifacts which occur for $m_Q a \sim 1$ in a conventional, propagating heavy quark approach, and which skewed the results of Ref. [3]. In Ref. [14], it is shown how such artifacts can be corrected for all $m_Q a$, order by order in ap (where p is a typical 3-momentum) and $\alpha_s(q^*)$ (where q^* is a scale of the order of a^{-1}). The most important of these correction is the so-called "EKM" factor. Without this factor, the the heavy-light decay constants become inconsistent with the static-light decay constants at fixed a once $m_Q a \gtrsim 0.5$. [15] Refs. [16,10] do not make this correction. However, the rather large statistical errors mask somewhat the inconsistency, and the final results, which more or less average between

Table 1: Examples of decay constant calculations from the "middle ages:" no major problems, but incomplete study of systematic errors. All of the calculations were performed in the quenched approximation, and none of them tried to estimate quenching errors. The other concerns listed are: (A) no continuum extrapolation, (B) inconsistent propagating and static quark normalization, (C) small physical volumes, (D) static approximation only, (E) $\mathcal{O}(1/M^2)$ operators neglected, (F) complete 1-loop perturbative correction not included, (G) lattice spacing effects on light quarks not estimated.

Group	f_B **(MeV)**	**Concerns**
ELC [16]	205(40)	A, B
UKQCD [17]	$160(^{+6}_{-6})(^{+59}_{-19})$ $176(^{+25}_{-24})(^{+33}_{-15})$	A
BLS [15]	205(40)	A
PCW [10]	180(50)	B, C
FNAL [9]	$188(23)(15)(^{+26}_{-0})(14)$	D
APE [18]	180(32)	A
NRQCD [11]	183(32)(28)(16)	E, G
Hiroshima [12]	184(7)(5)(37)(37)	F, G

the propagating and static quark results, are not unreasonable.

1.3 Modern Era

I place in the "modern era" those computations which make a serious attempt to quantify all the sources of systematic error, at least within quenched approximation. In addition, one of these calculations [19] tries to estimate the error due to quenching by comparing some quenched results with those on comparable lattice that include virtual quark effects. However the resulting estimates remain rather crude at the moment — see below. The three on-going computations which I place in the modern category are listed in Table 2, along with some representative results and my attempt to average them. It must be emphasized that the results from all three groups are still preliminary. I note that the recent computation by the APE group [18] could also be considered "modern" by many criteria. However, I have placed it in the previous section because, with only two values of the lattice spacing, it is difficult to draw any conclusions about the size of the systematic error of the continuum

Table 2: Decay constant calculations from the "modern era." All of the calculations are preliminary. The first error in all cases is statistical; the second represents systematic errors within the quenched approximation. The third error from MILC is an estimate of the quenching effect; the other groups do not consider quenching errors at this time. The systematic errors for JLQCD do not yet include the effects from the continuum extrapolation, estimated at "a few percent," and the errors from weak coupling perturbation theory, estimated as 5%. "BraveWA" are my brave (or foolhardy!) attempts at world averages, which ignore certain systematic inconsistencies in the data (see text), and assume that the quenched approximation results can be *corrected* by the MILC quenching error when this has a definite sign. The errors in world averages are combined total errors; a more conservative approach would result in errors larger by a factor of about 1.5.

Group	f_B **(MeV)**	f_{B_s}/f_B	f_{D_s} **(MeV)**
MILC [19]	$153(10)(^{+36}_{-13})(^{+13}_{-0})$	$1.10(2)(^{+5}_{-3})(^{+3}_{-2})$	$199(8)(^{+40}_{-10})(^{+10}_{-0})$
JLQCD [21]	$163(12)(^{+13}_{-16})$	–	$213(11)(^{+12}_{-18})$
FNAL [22]	166(10)(28)	1.17(4)(3)	215(7)(30)
BraveWA	175(30)	1.14(5)	221(25)

extrapolation.

2 Systematic Errors

The sources of the three largest systematic errors affecting the calculation of the MILC collaboration [19] are the extrapolation to the continuum, the "chiral" extrapolation (extrapolation from the lattice light — u and d — quark masses to the physical light quark masses), and the quenched approximation. Here, I will briefly discuss each of these systematic errors. (For a more comprehensive discussion, see Ref. [19].) As will become clear below, the dominant sources of error in the other two "modern" calculations may be different.

2.1 Continuum Extrapolation

With the Wilson fermions used in the MILC computation, the leading errors as $a \to 0$ are $\mathcal{O}(a)$, although the coefficient of a, as well as the size of higher order terms, are of course unknown *a priori.* In Fig. 1, I show this extrapolation for f_B and the ratio f_{B_s}/f_B. Assuming linearity in a over the full range of lattice spacings studied gives the central values. Another plausible assumption is that for the three smallest spacings (corresponding to $\beta = 6.0$, 6.3, and 6.52), terms of $\mathcal{O}(a)$ and higher are already quite small. This leads to a constant fit to these three points, and we take the difference between the two fits as one measure of the systematic error. Other measures of the error are obtained by comparing the linear fit to all points to a *linear* fit to the three points with smallest lattice spacing, and by varying or omitting the EKM corrections and repeating the linear fit to all points. (For more details see Ref. [19].)

For all the decay constants as well as almost all ratios of decay constants (the exception is f_B/f_{D_s}), the difference of linear and constant fits, as in Fig. 1, gives the largest error estimate and is therefore taken to be the systematic error of the continuum extrapolation. For the decay constants themselves, this error is rather large ($\sim$12–27%), reflecting the fact that the slope in a is steep. The ratios of decay constants are much better behaved with an error of $\sim$4–5%. These errors are in general the largest of all the systematic errors, for both decay constants and ratios.

The continuum extrapolation errors can be reduced by "improving" the action — removing discretization errors order by order in a by adding additional operators to the action (and correcting the axial current). New data with improved actions has appeared recently.[21,22] The preliminary results are

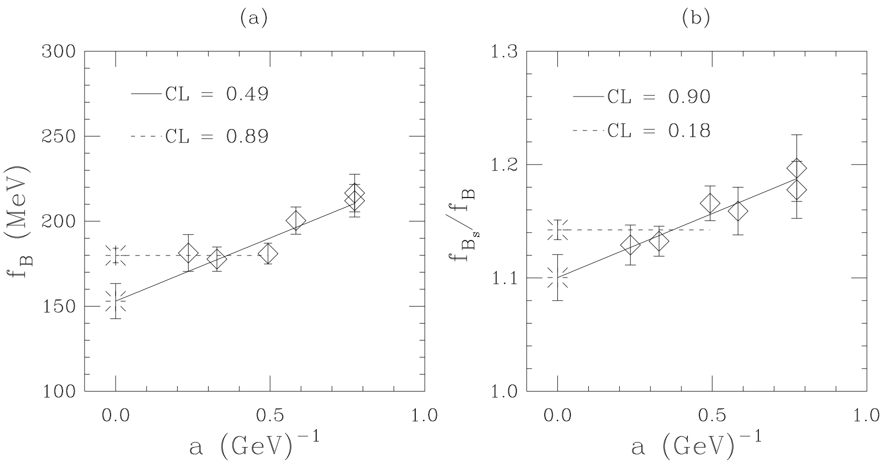

Figure 1: (a) Preliminary quenched results from the MILC collaboration for f_B as a function of lattice spacing a. The errors bars reflect only statistical errors and the errors from isolating the state of interest. The diamonds are the lattice data points, and the bursts are extrapolated values. The linear fit to all the diamonds (solid line) gives the central value; a constant fit to the three smallest lattice spacings (dotted line) gives one way to estimate the systematic error of the extrapolation. "CL" denotes confidence level. (b) Same as (a), but for the ratio f_{B_s}/f_B *vs.* a.

shown in Fig. 2.

FNAL implements the full EKM program [14] at $\mathcal{O}(a)$ and up to tadpole improved tree-level in α_s. JLQCD improves the action with "clover" (Sheikholeslami-Wohlert) fermions,[23] and adjusts the normalization and shifts from pole to kinetic mass *à la* EKM. [14] As I understand it, the main difference from the full EKM program at this order is in the corrections to the axial current, but these corrections are expected [2] to have little effect on the final results.

In practice, the apparent excellent agreement between the improved results of JLQCD and FNAL seen in Fig. 2 is somewhat misleading, since JLQCD uses m_ρ to set the scale, and FNAL, f_π. Indeed, when JLQCD uses f_π to set the scale, they see steeper a dependence. [2] Some time will thus be needed until the dust settles and the situation is clearer. However it seems quite obvious from Figs. 2 and 1 that the extrapolation to the continuum will be much better controlled in the improved case than in the Wilson case. One will be able to reduce the error still further by demanding equality between the results of the two methods.

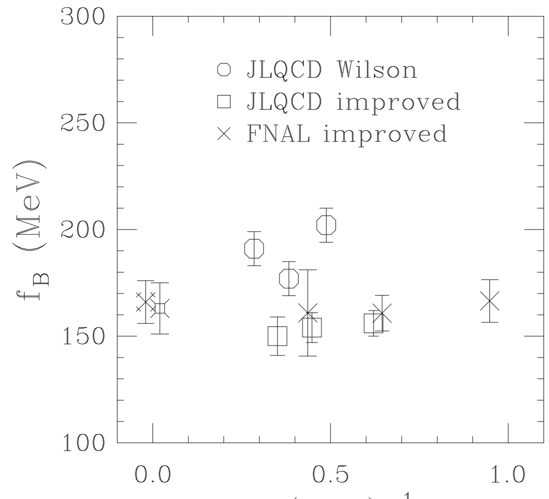

Figure 2: Preliminary quenched results from the JLQCD and FNAL collaborations for f_B as a function of lattice spacing a. The errors shown are statistical only. For JLQCD, results with both Wilson (octagons) and the improved action (squares) are shown; FNAL (crosses) uses improved action exclusively. Final extrapolated values with statistical errors are also shown: "fancy square" for JLQCD and "fancy cross" for FNAL.

2.2 *Chiral Extrapolation*

The lattice computations are typically performed with light quark masses (m_q) in the range $m_s/3 \lesssim m_q \lesssim 2m_s$. This is because using physical m_u, m_d would (a) require too much computer time, (b) require too large a lattice, and (c) introduce spurious quenching effects.[24] The "chiral extrapolation" is then the extrapolation in m_q to physical $m_{u,d}$. One must extrapolate not only the heavy-light decay constants and masses, but also, in general, one or more experimentally known light-light quantities in order to set the scale of the lattice spacing and determine the correct lattice light quark mass. Typical light-light quantities needed are m_π and either f_π or m_ρ.

The chiral extrapolation is a significant source of error, with the majority of the error coming from the light-light quantities. This is because the light-light quantities are more non-linear in quark mass. For example, although lowest order chiral perturbation theory predicts that m_π^2 is a linear function of quark mass, one sees, at the current statistical level, small but significant deviations from linearity that are not well understood.

The deviations from linearity could be due to unphysical effects such as the finite lattice spacing or the residual contamination by excited states. Even the more "physical" cause (chiral logs or higher order analytic terms in chiral perturbation theory) are a source of spurious effects because quenched chiral logs are in general different from those in the full theory.[24]

For the above reasons, the MILC collaboration presently fits quantities like m_π^2 to a linear form, despite the poor confidence levels. The systematic error is estimated by repeating the analysis with quadratic fits. The latter are constrained fits, since unfortunately the computation has been done with only three light quark masses. The systematic thus determined is $\leq 10\%$ for decay constants on all quenched data sets used to extrapolate to the continuum; usually it is $\lesssim 5\%$. After extrapolation to the continuum, the error is larger: 7% to 15%.

I emphasize that our reasons for choosing linear chiral fits for the central values are somewhat subjective, and it is possible that we will switch to quadratic fits in the final version of the work. To help us make the choice, we are studying a large sample of lattices at $\beta = 5.7$, with large volumes up to 24^3. On this sample we have six light quark masses and have mesons with nondegenerate as well as degenerate quarks. Should the switch be made, it would raise the central value for f_B by 23 MeV, that for f_{D_s} by 14 MeV, and other decay constants by comparable amounts. The systematic error within the quenched approximation would then become much more symmetric, with the continuum extrapolation the dominant positive error and the chiral extrapolation the dominant negative one.

It is worth noting here that the MILC and JLQCD are quoting rather different (by $\sim 15\%$) lattice scales from f_π at the one coupling ($\beta = 6.3$) they have in common. Although both groups use linear chiral fits, the chiral extrapolation may explain at least part of the discrepancy, since the masses of MILC's light quarks extend to higher masses than those of JLQCD, and quadratic terms are therefore more likely to be important for MILC. Another possible cause is finite size effects (JLQCD's lattice at $\beta = 6.3$ has size 32^3; MILC's, 24^3). However, the MILC study (mentioned above) of various volumes at $\beta = 5.7$ shows no significant finite size effects at the relevant physical volumes, and appears to give an upper bound or such effects that is smaller than what is needed to explain the discrepancy. The groups are in the process of comparing raw numbers and normalizations to try to settle this issue.

2.3 Quenching

The quenched approximation has one great advantage: it saves an enormous amount of computer time. However, it is not a true approximation, since there is no perturbative expansion of the full theory for which the quenched approximation is the first term. Thus one should think of it only as a model, and it is imperative to estimate its errors, and ultimately to move beyond it.

To this end MILC has repeated the computations on lattices with virtual quark loops included. I emphasize however that such computations are not yet "full QCD." This is because (1) the virtual quark mass is fixed and not extrapolated to physical up or down mass (2) the virtual quark data is not yet good enough to extrapolate to $a = 0$, and (3) there are two light flavors, not three. Thus the virtual quark simulations are used at this point only for systematic error estimation, and the error estimate so obtained must be considered rather crude.

Figure 3 shows one way the quenching error is estimated. We compare the smallest-a virtual quark simulation (the cross at $a = 0.47$ $(\mathrm{GeV})^{-1}$) with the quenched simulations, interpolated to the same value of a. See Ref. [19] for more details.

3 Brief remarks on the B parameters

Recent lattice computations [25,26] of B_{B_d} with propagating heavy quarks are consistent with older work. [3,27,16] They all give $B_{B_d}(5\mathrm{GeV}) \approx 0.9$, with errors (within the quenched approximation) of about 10%. However, the results with static heavy quarks [28,29,30] vary widely, from about 0.5 to 1.0. The variation appears to be due in large part to the large uncertainties in the perturbative corrections, in the matching both of lattice to continuum and of static effective theory to full theory. In addition, the poor signal-to-noise properties of the static theory also may lead to some difficulties. (For compilations of both propagating heavy-light and static-light results, see Refs. [29,30].)

MILC has recently begun a computation of static-light B parameters. Our very preliminary results give $B_{B_d}(5\mathrm{GeV}) \approx 0.9$. This is consistent with the results of the Kentucky group,[29] which is not surprising since the perturbative corrections are treated in the same way.

None of the existing calculations (propagating or static) is fully "modern" in the sense of Section 1, since the extrapolation to the continuum has not yet been studied carefully. However, the indications are that B_{B_d} has only mild dependence on the lattice spacing. [25] If I also assume that the Kentucky approach to the static-light perturbative corrections is the most "reasonable" since it gives results that agree with the propagating heavy-light results, I arrive at $B_{B_d}(5\mathrm{GeV}) = 0.86(3)(10)(8)$ for my (prejudiced) world average. Here the first error is statistical, the second summarizes systematics within the quenched approximation, and the third is my *guess* of the quenching error. The renormalization group invariant, $\hat{B}_{B_d}$ at next to leading order is then $\hat{B}^{nlo}_{B_d} = 1.37(22)$. Combining this with the "brave world average" (Table 2) for

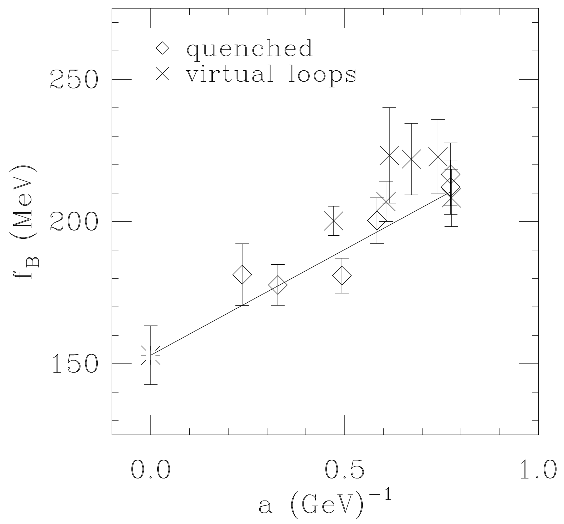

Figure 3: Preliminary MILC results for f_B as a function of lattice spacing used for estimating the effects of quenching. The diamonds are the same as in Fig. 1(a). The crosses (virtual quark loops included) are not extrapolated to the continuum.

f_B gives

$$\xi_d \equiv f_B \sqrt{\hat{B}^{nlo}_{B_d}} = 205(39)\ \text{MeV}. \tag{1}$$

The ratio B_{B_s}/B_{B_d} is considerably more stable. All groups find it to be very close to 1. Including the preliminary MILC results (on lattices with and without virtual quarks), I find a world average $B_{B_s}/B_{B_d} = 1.00(1)(2)(2)$. Combining this with the "brave world average" for f_{B_s}/f_B gives

$$\xi^2_{sd} \equiv \frac{f^2_{B_s} B_{B_s}}{f^2_B B_{B_d}} = 1.30(12) \tag{2}$$

I emphasize that the errors in Eqs. (1) and (2) are based on the optimistic assumptions that go into the "brave world averages." A more conservative approach would result in errors larger by a factor of about 1.5.

I note that it is also possible to calculate ξ^2_{sd} directly,[31] without calculating decay constants and B parameters separately. We are getting (still preliminary) $\xi^2_{sd} = 1.68(10)^{+43}_{-38}$. Given the large errors, this is consistent with Eq. (2). The direct method has, however, several advantages, and ultimately may be competitive with or even superior to the separate computations.

4 The "Post-Modern Era"

I expect a significant improvement over the next few years in the lattice evaluation of decay constants. Within the quenched approximation, there should be good control of all the systematic errors. In particular, improved actions (and possibly smaller lattice spacings) will allow for a considerable reduction of the continuum extrapolation error. Further, the chiral extrapolations should be better controlled by using smaller quark masses on physically larger lattices. (JLQCD has already been able to move in this direction.) Within two or three years, the conservative total error within the quenched approximation should be $\sim 10\%$ on decay constants, $\sim 3\%$ on the ratios f_{B_s}/f_B and f_{D_s}/f_D, and $\sim 5\%$ on the ratios f_B/f_{D_s} and f_{B_s}/f_{D_s}. (The latter ratios — see Ref. [19] for current values — may prove the best way to get at f_B and f_{B_s}, especially if a future Tau-Charm factory allows for precise experimental determination of f_{D_s}.[32])

The quenching errors on the decay constants will be more difficult to control, but I expect a gradual improvement in the estimates of this error. As discussed above, the current estimates are rather crude, so I emphasize that "improvement in the estimates" does not necessarily mean "reduction of the error!" In the MILC collaboration, we expect to have a first extrapolation of virtual quark results to $a \to 0$ within one or two years.

Calculations of the B parameters are less far along, and it is therefore more difficult to predict what the improvement will be over the next few years. I am hopeful that studies of the lattice-spacing dependence and of virtual quark effects, as well as perhaps higher order perturbative calculations and/or non-perturbative evaluations of the renormalization constants, will result in errors on B_{B_d} of $\sim 5\%$ within the quenched approximation and another $\sim 5\%$ due to quenching. For the ratio B_{B_s}/B_{B_d}, which is already quite well determined, one can hope to reduce the continuum extrapolation and quenching errors to 1%.

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