Heavy quark physics on the lattice

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I review the current status of lattice calculations of the properties of bound states containing one or more heavy quarks. Many of my remarks focus on the heavy-light leptonic decay constants, such as f_B, for which the systematic errors have by now been quite well studied. I also discuss B-parameters, semileptonic form factors, and the heavy-light and heavy-heavy spectra. Some of my "world averages" are: $f_B = 200(30)$ MeV, $f_B\sqrt{\hat{B}_{B_d}} = 230(40)$ MeV, $f_{B_s}/f_B = 1.16(4)$ and $f_{B_s}\sqrt{\hat{B}_{B_s}}/f_B\sqrt{\hat{B}_{B_d}} = 1.16(5)$.

1. INTRODUCTION

Many of the parameters of the Standard Model can be constrained by measurements of the properties of hadrons containing heavy quarks. To take advantage of such experiments, however, one needs theoretical determinations of the corresponding strong-interaction matrix elements. Lattice gauge theory provides, at least in principle, a means of computing hadronic matrix elements with control over all sources of systematic error. Here, I review the current status of these computations.

A review like this is necessarily somewhat idiosyncratic in the topics it covers and the time spent on each. I devote a large fraction of my time to f_B. Because lattice data for leptonic decay constants is more extensive than for any other heavy quark quantity, I am able to discuss in detail several vexing issues: renormalon effects, systematic errors due to large lattice quark masses, extrapolation to the continuum, and quenching errors. I then turn to the "B-parameters," particularly B_{B_d} and B_{B_s}, and to semileptonic form factors for $B \to \pi$. Finally, I discuss spectra of heavy-light and heavy-heavy hadrons. I conclude with a few remarks on the unitarity triangle.

Heavy quark masses, which are included in Lubicz's talk at this conference [1], are examined here only for the light they shed on renormalon effects. The heavy quark potential is discussed in detail by Bali [2], and I omit it here. Recent reviews of heavy quark physics on the lattice by Hashimoto [3] and Draper [4] complement the current treatment.

2. LEPTONIC DECAY CONSTANTS

Table 1 and Fig. 1 show recent results for f_B; while Fig. 2 plots the quenched f_B data as a function of lattice spacing a. One notices immediately that the UKQCD [13,14] and CP-PACS results [17,18] (with heavy clover quarks and with NRQCD) are somewhat high compared to those of the other groups. Furthermore, the errors given by CP-PACS are rather small. Does this mean one should increase the central value and lower the error significantly from the "world averages" for quenched f_B quoted recently by and Hashimoto [3] ($f_B = 170(20)$ MeV) or Draper [4] ($f_B = 165(20)$ MeV)?

In the case of UKQCD, a major cause of the quite high result is the choice of the r_0 parameter [15] from the static potential to set the scale. Since ar_0 can be extracted easily and precisely from lattice data, it provides an excellent way to check the scaling of physical quantities as a is changed. However, I would argue that because r_0 is not directly determined by experiment, but only through phenomenology, ar_0 should not be used to set the absolute scale of a. Indeed, Sommer [15] remarks that an error of roughly 10% should be associated with his value $r_0 = 0.5$ fm.

Of course, in the quenched approximation, the scale set by different quantities will differ. But the experimental uncertainty in r_0 introduces an unnecessary additional error. Indeed, while scales set by various directly measurable quantities, *i.e.*, f_π [14], m_ρ [13], f_K and m_K [19], or m_{K^*} and m_K [12], differ by at most 7% at $\beta = 6.2$, the r_0 scale differs from these by as much as 15%. The

Quenched	f_B (MeV)	f_{B_s}/f_B
FNAL97 [5]	$164(^{+14}_{-11})(8)$	$1.13(^{+5}_{-4})$
APE97 [6]	$180(32)$	$1.14(8)$
JLQCD98 [7]	$173(4)(12)$	$\simeq 1.15$
MILC98 [8]	$157(11)(^{+25}_{-9})(^{+23}_{-0})$	$1.11(2)(^{+4}_{-3})(3)$
Ali Khan98 [9]	$147(11)(^{+8}_{-12})(9)(6)$	$1.20(4)(^{+4}_{-0})$
JLQCD99 [10]	$167(7)(15)$	$1.15(3)(1)(^{+3}_{-0})$
APE99 [11]	$179(18)(^{+34}_{-9})$	$1.14(2)(1)$
APE00 [12]	$174(22)(^{+7}_{-0})(^{+4}_{-0})$	$1.17(4)(^{+0}_{-1})$
UKQCD00 [13,14]	$218(5)(^{+5}_{-41})$	$1.11(1)(^{+5}_{-3})$
MILC00* [16]	$173(6)(16)$	$1.16(1)(2)$
CP-PACS00 [17,18]	$188(3)(9)$	$1.148(8)(46)(^{+39}_{-0})$
CP-PACS00(NR)* [17]	$191(5)(11)$	$1.150(9)(6)$
Lellouch&Lin00 [19]	$177(17)(^{+22}_{-26})$	$1.15(2)(^{+3}_{-2})$
$N_f = 2$	f_B (MeV)	f_{B_s}/f_B
Collins99(NR) [20]	$186(5)(19)(9)(13)(^{+50}_{-0})$	$1.14(2)(^{+0}_{-2})$
MILC00* [16]	$191(6)(^{+24}_{-18})(^{+11}_{-0})$	$1.16(1)(2)(2)$
CP-PACS00 [17,18]	$208(10)(11)$	$1.203(29)(43)(^{+38}_{-0})$
CP-PACS00(NR)* [17]	$204(8)(15)(^{+44}_{-0})$	$1.179(18)(7)$

Table 1
f_B and f_{B_s}/f_B from various groups. Computations using NRQCD are explicitly noted with (NR); all others use relativistic formalisms. A * denotes a preliminary result.

UKQCD result $f_B = 218(5)^{+5}_{-41}$ MeV [13] becomes $195(8)^{+0}_{-13}$ MeV [14] when f_π is used to set the scale. Other systematic errors (some of which are discussed below) then easily account for any remaining difference between this result and those of other groups.

Turning now to the CP-PACS results [17,18], we see from Fig. 2 that these are from rather coarse lattices ($a > 0.57(\mathrm{GeV})^{-1}$) and that the lattice spacing dependence is quite large. Although the Iwasaki gauge action [22] used here generally gives quite small scaling violations, decay constants are an exception [23]. In particular, f_π has an a-dependence that is qualitatively very similar to that of f_B. Indeed, in the heavy-clover case, f_B/f_π scales considerably better than f_B/m_ρ and gives $f_B = 174(4)(3)$ MeV instead of the quoted 188(3)(9) MeV. Similarly, with NRQCD, f_B/f_π gives $f_B = 169(7)$ MeV instead of the quoted 191(5)(11) MeV. In the absence of data at smaller a, I believe the f_B/f_π results are more reliable. Furthermore, the discrepancy suggests that the systematic error estimates should be increased. The CP-PACS results are then completely compatible with the average from other groups and do not imply a significant increase in central value or decrease in error for quenched f_B.

Before determining a world average for f_B, I now examine several other sources of systematic error that affect calculations by various groups.

2.1. Renormalon Shadows

In lattice HQET [24], NRQCD [25], and the Fermilab [26] formalisms, corrections to the static limit require the addition of higher dimensional operators, with coefficients that depend on the heavy quark mass. Such power corrections are most easily discussed in HQET. The framework I will use is that of Martinelli and Sachrajda [27]; for additional relevant discussion see Ref. [28].

In lattice HQET a physical quantity $\mathcal{P}$, which depends on the heavy quark mass m_Q, can be expressed through order $1/m_Q$ as

$$\mathcal{P}(m_Q) = C_1\langle|O_1|\rangle_a + \frac{C_2}{2m_Q}\langle|O_2|\rangle_a \qquad (1)$$

where the short distance coefficients C_1 and C_2

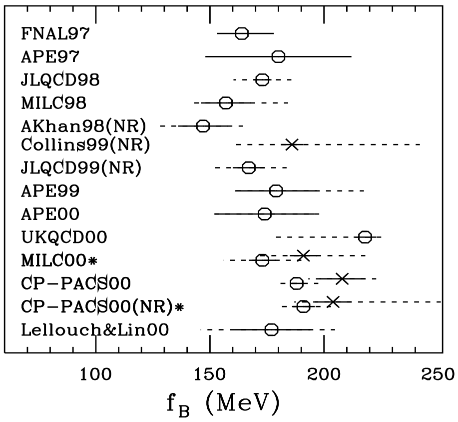

Figure 1. Results for f_B from various groups in the quenched approximation (circles) and with $N_f = 2$ (crosses). Solid lines show statistical errors; dashed lines show systematic and statistical errors, added in quadrature. A * denotes a preliminary result. Calculations using NRQCD are marked (NR); all other computations using relativistic formalisms. For references see Table 1.

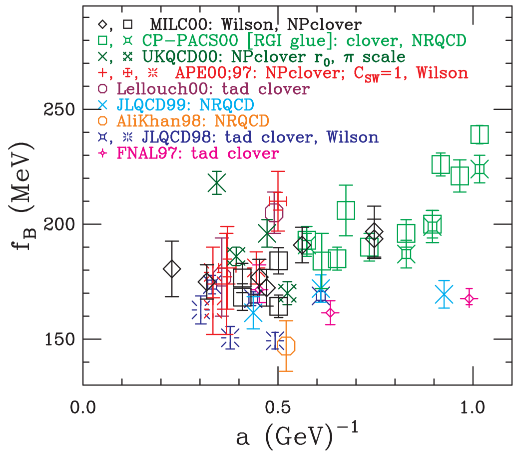

Figure 2. Recent world data for f_B in the quenched approximation *vs.* lattice spacing. Only statistical errors are shown. For references see Table 1.

are perturbatively calculable functions of am_Q and the coupling g. The higher dimension operator O_2 mixes with O_1 with a power divergent (like $1/a$) coefficient. In particular, we have

$$\begin{aligned}
C_1(am_Q, g) &= c_1(am_Q, g) + \frac{\tilde{c}_1(am_Q, g)}{2am_Q} \\
c_1(am_Q, g) &= 1 + \#g^2 + \#g^4 + \ldots \\
\tilde{c}_1(am_Q, g) &= \#g^2 + \#g^4 + \ldots \\
C_2(am_Q, g) &= 1 + \#g^2 + \#g^4 + \ldots\ , \qquad (2)
\end{aligned}$$

where "#" stands for logs of am_Q or constants. For future reference, I denote $C_2O_2/(2m_Q)$ as the "bare" $1/m_Q$ operator. Terms in C_1O_1 generated by loop corrections to the bare $1/m_Q$ operator I call the "subtraction." (The subtraction is just $[\tilde{c}_1/(2am_Q)]O_1$ in the HQET case.) Finally, the sum of the bare $1/m_Q$ operator and the subtraction gives the "renormalized" $1/m_Q$ operator.

It is widely believed, but not rigorously proven, that the series c_1 and $\tilde{c}_1$ have renormalon ambiguities at $\mathcal{O}(\Lambda_{\mathrm{QCD}}/m_Q)$. In the sum C_1, these ambiguities should cancel. However, since the cancellation only occurs at high order, one might anticipate [27] that the low order series for C_1 "converges" rather slowly. I call this effect the "renormalon shadow:" although the renormalons are formally gone, their influence lingers. We expect renormalon shadows in any lattice calculation where power law divergences are subtracted perturbatively.

As an example, consider the lattice HQET computation of the b quark mass. In the static limit, m_b is just the meson mass M_B. A nontrivial calculation must therefore include the $1/m_Q$ correction, *i.e.*, the term of $\mathcal{O}(\Lambda_{\mathrm{QCD}})$. The relevant coefficients were calculated to two loops and combined with lattice data in [29]. The "bad news" from this calculation is that the perturbative error on m_b that would result from stopping at one loop is $\sim$300 MeV, namely the size of the effect one is trying to compute. The rather large error can be taken as evidence that the renormalon shadow is real in this case. However, I also find "good news" in Ref. [29]: The $\sim$ 300 MeV one-loop error could have been correctly estimated by a standard one-loop analysis. Such analysis involves computing the scale q^* [30] (here $q^* = 1.446/a$), and seeing how the one-loop result changes under reasonable variations in q^* (say $1 \leq aq^* \leq \pi$) and in coupling schemes (say $\alpha_{\overline{MS}}(q^*)$ or $\alpha_V(q^*)$).

Now let us apply what we have learned to the calculation of f_B in NRQCD. Paralleling Eq. (1), the NRQCD expression for f_B takes roughly the form

$$f_B \propto C_1 \langle 0|O_1|B\rangle_{a,m_Q} + \frac{C_2}{2m_Q}\langle 0|O_2|B\rangle_{a,m_Q} \ , \quad (3)$$

where

$$O_1 = \bar{q}\gamma_0\gamma_5 Q \ ; \qquad O_2 = \bar{q}\gamma_0\gamma_5 \not{D}_\perp Q \ , \quad (4)$$

with Q the heavy quark field, q the light quark field and $\not{D}_\perp \equiv \sum_{i=1}^{3}\gamma_i D_i$ the spatial Dirac operator. Note that, unlike in Eq. (1), the matrix elements in Eq. (3) depend explicitly on m_Q. This is because m_Q already appears in the lowest order NRQCD Lagrangian.

As in the HQET case, O_2 has a $1/a$ divergence proportional to O_1.[1] Therefore one expects that a renormalon shadow will appear, producing large errors in the renormalized $1/m_Q$ matrix element unless the perturbative calculation is taken to high order. However, because the matrix element of O_1 also depends on m_Q, the renormalized O_2 matrix element in NRQCD is not the entire $\mathcal{O}(1/m_Q)$ effect. This is a crucial difference from the HQET case. The importance of the renormalon shadow here is thus a numerical question.

The discussion of f_B in the Fermilab formalism is similar. In Eq. (3), one just replaces $1/(2m_Q)$ by ad_1, where d_1 is the coefficient of the "rotation" defined in [26] and is a function of am_Q. As $am_Q \to \infty$, $d_1 \to 1/(2am_Q)$, thereby reproducing NRQCD. Other the other hand, $d_1 \to 0$ as $a \to 0$. This means that the subtraction here does not diverge (and in fact vanishes) as $a \to 0$. Note however that the relative perturbative uncertainty in the renormalized $1/m_Q$ operator is unaffected by the presence of ad_1, which is just an overall factor. O_2 itself is still power divergent and a renormalon shadow should still appear. Still, the factor ad_1 may reduce the numerical importance of the shadow.

To discuss the renormalon shadow in f_B quantitatively, I start with the Fermilab formalism, using MILC data [16] at $\beta = 6.15$ with the non-perturbative [31] value of the clover coefficient C_{SW}. The perturbative corrections have been calculated [32] at one loop. I define the subtraction here as O_1 times the difference between its complete perturbative coefficient and the coefficient when O_2 is omitted from the calculation [33]. Since q^* has not been computed for this quantity, I choose the value that comes from the static-light Z_A [34] ($q^* = 2.85/a$) and use $\alpha_V(q^*)$ [30]. To get the uncertainty in the subtraction, I then replace $q^* \to 1/a$. (The change is smaller when q^* is increased, even to $10/a$, or when the scheme is changed to $\overline{\rm MS}$ or "boosted perturbation theory" [35].) The ratio of the uncertainty to the renormalized $1/m_Q$ matrix element is then quite large: 60%. I take this as evidence that a renormalon shadow is indeed present.

However, since the renormalized $1/m_Q$ operator contributes only a small amount to f_B, the ultimate effect of the the renormalon shadow is small. Figure 3 shows the separate effects of the bare $1/m_Q$ operator and the subtraction, as well their combined effects (the renormalized $1/m_Q$ operator). Clearly, most of f_B, as well as its m_Q dependence, comes from the matrix element of O_1. The matrix element of the bare $1/m_Q$ operator adds 4.6% to f_B; while the subtraction is -2.6%. The renormalized $1/m_Q$ operator thus contributes only $(2.0 \pm 1.2)\%$, where I've included the uncertainty from the renormalon shadow. The uncertainty (~ 2 MeV) is smaller than many other systematic effects in current computations. Note that the error which would be made from including the bare $1/m_Q$ operator without the one-loop subtraction is probably larger than leaving out the operator completely.

In the NRQCD case, most of the numerics (for f_{B_s} at $\beta = 6.0$) can be found in Ref. [36]. Here the bare $1/m_Q$ contributes $\sim 11\%$; the one-loop subtraction, $\sim -7\%$. The inclusion of the renormalized $1/m_Q$ operator is thus a $(4 \pm 4)\%$ effect, where the error again comes from my varying q^* in the subtraction. It is no surprise that the effect and error here are larger than in the Fermilab case: $1/(2am_Q)$ is considerably larger than d_1 for relevant values of am_Q. However, the 4% uncertainty (~ 7 MeV) is still smaller than several

[1] Note, however, that the separation of C_1 into c_1 and $\tilde{c}_1$ is now no longer meaningful since the since the loop diagrams can generate arbitrary functions of aM.

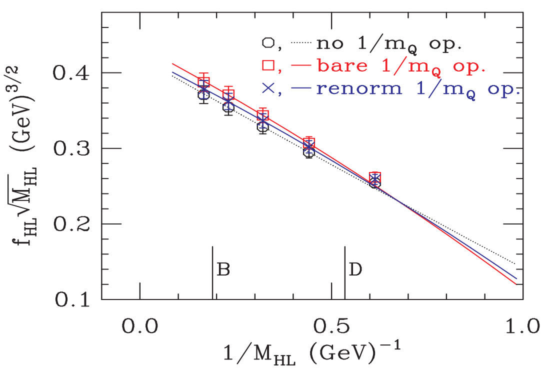

Figure 3.
Effect of bare and renormalized $1/m_Q$ operator on heavy-light decay constants in the Fermilab formalism. M_{HL} is the mass of a heavy-light pseudoscalar meson, and f_{HL} is its decay constant. MILC data ($\beta = 6.15$, $C_{SW} = 1.644$) [16] is used.

other systematic effects.

My conclusion is that the calculation of leptonic decay constants of B mesons is under control for both the Fermilab formalism and NRQCD, despite the presence of renormalon shadows. The issue however needs to be considered on a case-by-case basis. There is no guarantee that renormalon shadows are negligible for other physical quantities, systems, or orders in perturbation theory. Indeed, in many computations the perturbation theory is only put in at (tadpole-improved) tree level. The subtractions are therefore left out entirely. From the above discussion it seems likely that the inclusion of higher dimension operators in such a situation is worse them omitting them entirely, and may lead to large renormalon shadow uncertainties (see Sec. 5).

Two further comments: (1) My numerical estimates of renormalon shadow uncertainties are completely standard and do not involve the details of the renormalons at all. Therefore it is possible to take the term "renormalon shadow" as merely a convenient shorthand for the phrase "large perturbative uncertainties in power divergent subtractions." (2) The error I estimate for renormalon shadows is of course not the only perturbative uncertainty: many of the terms in C_1 would be present even if O_2 were omitted. Once it is ascertained that renormalon shadows are under control, it is probably preferable to quote simply an overall perturbative uncertainty, obtained by varying q^* everywhere. At the current state of the art (one loop) the overall perturbative uncertainty is generally larger than what I estimated above for the renormalon shadow. At high enough order, my method would probably overestimate the perturbative uncertainty because I have not taken into account the expected cancellation of renormalons between c_1- and $\tilde{c}_1$-type terms.

2.2. Discretization Errors in MILC Data

In MILC calculations of quenched f_B [8] using Wilson and static heavy quarks, the most important source of systematic error was the continuum extrapolation. This error was manifest in the difference between an linear extrapolation of f_B from all data sets ($0.2\ \mathrm{GeV}^{-1} \lesssim a \lesssim 0.8\ \mathrm{GeV}^{-1}$) and a constant extrapolation from only the finer lattices ($a \lesssim 0.5\ \mathrm{GeV}^{-1}$; $\beta \geq 6.0$). As of June, 1999 (with additional running from what appeared in [8]), the former extrapolation gave $f_B = 154(11)$ MeV; while that latter, 180(10) MeV. On the basis of the behavior of f_{B_s}/f_B, we chose the linear extrapolation for the central value and took the large 26 MeV difference as the discretization error.

Since that time, two developments have significantly reduced the discretization error we quote. First of all, Tom DeGrand and I have recalculated [34] the one-loop scale q^* appropriate to the static-light axial current renormalization constant Z_A^{SL}. Instead of Hernandez and Hill's result $q^* = 2.18/a$ [37] for tadpole-improved light Wilson fermions, we find a q^* that is mildly dependent on am_Q (as Z_A^{SL} is) and is $\approx 1.4/a$ for typical values of am_Q. We differ from Ref. [37] because: (a) we define q^* using the complete integrand for Z_A^{SL} (including the continuum part, which gives the m_Q dependence), and (b) we do not discard pieces of the Z_A^{SL} integrand which vanish by contour integration — such pieces do not vanish for q^* because of the additional $\ln(q^2)$ factor. Difference (b) is responsible for most of the discrepancy.

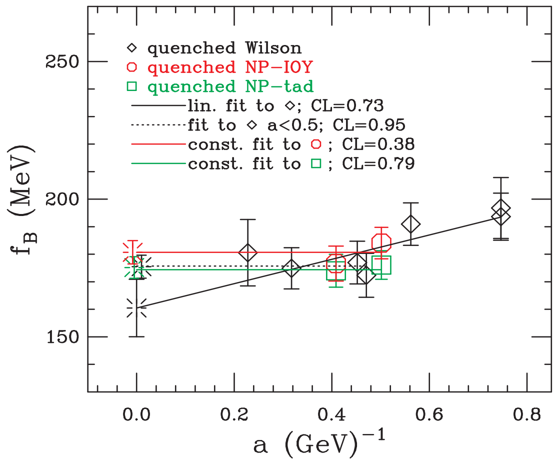

Figure 4.
Updated MILC data [16] for quenched f_B *vs.* lattice spacing. Wilson and nonperturbative clover ("NP") fermions with the Fermilab formalism are used. "NP-tad" and "NP-IOY" represent two different ways of renormalizing the heavy-light axial current — see text.

In the MILC calculation, the static-light q^* described above is employed as the central value of q^* not only in the static-light renormalization but also in the (propagating) heavy-light renormalization, for which Z_A, but not q^*, has been computed [38]. The new value $q^* \approx 1.4/a$ considerably reduces the a-dependence of f_B with Wilson quarks [16]. The difference between the two extrapolations of this data is now 15 MeV instead of 26 MeV (see Fig. 4).

In addition to reanalyzing the Wilson data, we recently completed running at $\beta = 6.0$ and 6.15 with nonperturbative [31] clover fermions and the full Fermilab formalism through order $1/m_Q$. The $1/m_Q$ operator is always renormalized at one loop [32,33], as in Sec. 2.1. For the rest of the renormalization of the heavy-light axial current, we use either the one-loop calculation [32], or a "tadpole renormalization" Ansatz designed to reproduce the nonperturbative renormalization of Ref. [31] at small mass and have a sensible limit as $m_Q \to \infty$. These two approaches are shown as "NP-IOY" and "NP-tad," respectively, in Fig. 4. NP-tad is described in more detail in Sec. 2.3. Both NP-IOY and NP-tad are correct to $\mathcal{O}(a)$. Higher order effects (*e.g.*, $\mathcal{O}(a^2)$, $\mathcal{O}(g^4)$) are complicated but, hopefully, rather small. Each approach is fitted to a constant; the difference in extrapolated value from the two NP approaches gives an estimate of these higher order effects.

From Fig. 4 it now seems clear that it was a mistake to choose only the linear extrapolation to find the central value of the Wilson data in the continuum: the linear fit gives the lowest extrapolated value of all four approaches. (In the case of f_{B_s}/f_B, the linear extrapolation differs even more from the other three fits.) MILC currently averages all four approaches for the central value and defines the discretization error as the standard deviation of the four. The result appears as the quenched "MILC00" point in Fig. 1 and Table 1.

2.3. "Nonperturbative" Heavy-Lights

APE [12,21], UKQCD [13,14], and MILC [16] have simulated heavy-light physics using nonperturbative clover fermions. The starting point is the expression for the improved, renormalized axial current A_0^R:

$$A_0^R = Z_A\sqrt{4\kappa_q\kappa_Q}(1+\frac{b_A a m_{Q,0}}{2})[A_0+c_A a\partial_0 P_5] \ ,$$
$$A_0 = \bar{q}\gamma_0\gamma_5 Q \ ; \qquad P_5 = \bar{q}\gamma_5 Q \ , \tag{5}$$

where κ_q, κ_Q are the light and heavy quark hopping parameters, respectively, and $m_{Q,0}$ is the bare heavy quark mass ($a m_{Q,0} \equiv 1/(2\kappa) - 1/(2\kappa_c)$). For simplicity, the light quark mass has been set to zero.

All three groups use the nonperturbative values of Z_A and c_A in Eq. (5) (as well as the clover coefficient C_{SW}) computed by the Alpha collaboration [31]. The groups differ, however, on the choice of the coefficient b_A, which was not computed nonperturbatively in Ref. [31]. At $\beta = 6.2$, for example, APE uses $b_A = 1.24$ from the one-loop calculation [40] and boosted perturbation theory [35]; while UKQCD uses $b_A = 1.47$ from a preliminary nonperturbative calculation of Bhattacharya *et al.* [39]. This difference accounts for any disagreement between APE and UKQCD results that remains once the scales are set in the same way. MILC's b_A is taken from perturbation theory [40], but with coupling $\alpha_V(q^*)$, with q^* chosen as the value ($\approx 1/a$) which produces the nonperturbative result [31] for the similar quantity b_V. This gives values of b_A (1.47 at $\beta = 6.15$ and 1.42 at $\beta = 6.0$) that are quite close to those of UKQCD.[2]

APE and UKQCD apply Eq. (5) directly for moderate values of $am_{Q,0}$ (up to ~ 0.5 at $\beta = 6.2$ and ~ 0.75 at $\beta = 6.0$). This allows them to reach a maximum meson (pole) mass of ≈ 2 GeV at $\beta = 6.2$, where I've set the scale by m_ρ. Using HQET, which implies that $f_{HL}\sqrt{M_{HL}}$ should be a polynomial in $1/M_{HL}$ (up to logs), they then attempt to extrapolate the results up to the B mass. (M_{HL} and f_{HL} are the mass and decay constant of a generic heavy-light pseudoscalar meson.)

There are of course systematic errors associated with this approach. First of all, one does not know *a priori* what order polynomial to use; there is a large difference ($\sim 10\%$) between the results of linear and quadratic extrapolations. A second systematic effect is more subtle. Although discretization errors in Eq. (5) are very small for $am_{Q,0} \ll 1$ and appear to remain quite small even for the maximum $am_{Q,0}$ used by ULQCD and APE, those errors grow rapidly with $am_{Q,0}$. Indeed, for $am_{Q,0} \to \infty$, A_R in Eq. (5) does not approach a static limit as it should, but goes to $-\infty$ because $c_A < 0$ and $\partial_0 P_5 \sim \sinh(M_{HL,1}) \sim am_{Q,0}$. (Here $M_{HL,1}$ is the meson pole mass.) Even if c_A were zero, A_R would still blow up because of the term $b_A am_{Q,0}$. Thus, small discretization errors may be magnified significantly by the extrapolation to the B.

To estimate the latter error, one may compare to the MILC "NP-tad" approach [42]. The MILC goal is to replace Eq. (5) by an expression that is equivalent through $\mathcal{O}(a)$ but which gives a sensible limit for $f_{HL}\sqrt{M_{HL}}$ as $am_{Q,0} \to \infty$. The result is then used at arbitrary $am_{Q,0}$, *à la* Fermilab. To do so, we first define

$$R(M_{HL}) \equiv \frac{\langle 0|\partial_0 P_5|HL\rangle}{m_{Q,0}\langle 0|A_0|HL\rangle}\ , \qquad (6)$$

where HL is a generic heavy-light pseudoscalar meson. Due to a cancellation of $\sinh(M_{HL,1})$ (from ∂_0) and the explicit $m_{Q,0}$ in the denominator, one expects R has a finite limit as $am_{Q,0} \to \infty$. This is confirmed by simulations. Then

$$A_0^R = Z_A\sqrt{4\kappa_q\kappa_Q}\sqrt{1+(b_A+2c_A R)am_{Q,0}}A_0 \qquad (7)$$

gives results for $\langle 0|A_0^R|HL\rangle$ that are identical to Eq. (5) through $\mathcal{O}(a)$. However, because $\kappa_Q am_{Q,0} \to 1/2$ as $\kappa_Q \to 0$, Eq. (7) has a static limit, unlike (5). Indeed, (7) is just a version of the Fermilab formalism at tadpole-improved tree level (the $1/m_Q$ operator is omitted for simplicity), but with a special (mass-dependent) value for the tadpole factor: $u_0 = (b_A + 2c_A R)^{-1}$. The similarity to tadpole improvement (within the context of nonperturbative renormalization) is the reason for the name "NP-tad."

Figure 5 shows the effect of reanalyzing the UKQCD results with the NP-tad normalization. This $\mathcal{O}(a^2)$ effect is 2% to 5.5% on their data points, but 8.5% after extrapolation to the B.

An additional difference between MILC and UKQCD or APE is that MILC, following the Fermilab formalism, defines the meson mass as the kinetic mass M_2, rather than the pole mass M_1. Since the difference in masses is $\mathcal{O}(a^2)$ this does not affect the $\mathcal{O}(a)$ improvement. M_2 is determined by adjusting the measured meson pole mass upward by $\Delta M \equiv m_2 - m_1$, where m_2 and m_1 are the heavy quark kinetic and pole masses, respectively, as calculated in tadpole-improved tree approximation [26], with the mean link fixed by κ_c. Figure 5 shows the effect of this shift on the UKQCD data (after first changing to NP-tad normalization). One can check the MILC determination of M_2 by using instead the M_2 values directly computed by UKQCD from the meson dispersion relation. The resulting curve is not shown in Fig. 5 because it is indistinguishable from the fit to the diamonds.

Note that the effects of the NP-tad norm and the shift by ΔM almost cancel, so that the final results of UKQCD and MILC are actually

[2] At $\beta = 6.0$ and 6.2, a new nonperturbative calculation of all the coefficients, including b_A, is now available [41]. It will be interesting to see how these values affect the results for f_B.

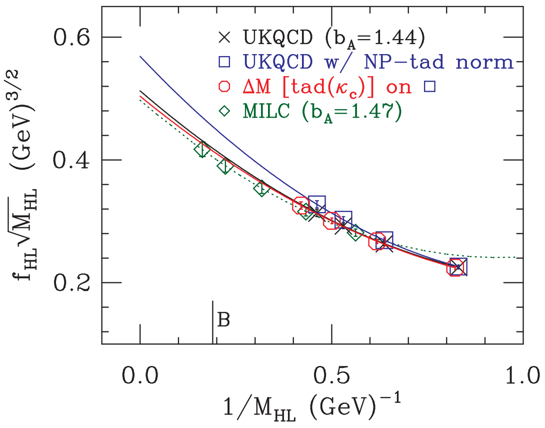

Figure 5. Effects, at $\beta = 6.0$, of various treatments of heavy-lights using nonperturbative clover fermions. UKQCD results (crosses) have been adjusted to same scale (from f_π) as MILC results (diamonds). Squares show the effect of adjusting the crosses using the NP-tad normalization. Octagons include the tadpole-improved mass shift (described in text) on the squares.

quite consistent as long as the scale is set in the same way. Since the two effects are logically independent, however, the 17 MeV difference for f_B among the curves in Fig. 5 is a measure of the discretization error. This error is still 15 MeV at $\beta = 6.2$. I would not, however, argue that this error should be added to those quoted by UKQCD. They already include a discretization error error obtained by comparing the results at $\beta = 6.2$ and 6.0. Indeed, the discretization error computed that way is 16 MeV (using an m_ρ scale), identical to the estimate above.

The goal of the approaches discussed in this subsection is to treat the heavy-lights in a way that takes advantage of the known nonperturbative renormalizations for light-lights. But since one either works in or extrapolates to a region where am_Q is not small, the systematic errors are not necessarily small. Furthermore, because the methods agree at $\mathcal{O}(a)$, there is no way *a priori* to distiguish among them at the level of Eq. (5). If one insists on using the nonperturbative information for the heavy-lights, then I prefer the NP-tad approach (and equating the physical mass with M_2) because (a) having a static limit seems desirable if we want to use (or extrapolate to) large am_Q, and (b) it appears to scale somewhat better: the difference between f_B at $\beta = 6.0$ and 6.15 is only 2 MeV (see Fig. 4). However, this is far from definitive: point (a) is subjective, and point (b) is not very convincing with only two lattice spacings.

The situation will improve soon. Although a true nonperturbative treatment of heavy-lights seems difficult, a nonperturbative computation of the $\mathcal{O}(a)$ renormalization/improvement coefficients for the *static*-light axial current is close to completion [43]. This will be an important advance, since it will allow one to compute f_B with an interpolation between two nonperturbative calculations. The extrapolation error of the UKQCD or APE approach will therefore be much reduced. One will also be able to see how well NP-tad is really doing in the intermediate region.

A alternative computation with nonperturbative clover fermions that can be done today is the “NP-IOY” approach mentioned in the previous section. However, since it is just a marriage of standard techniques, it is perhaps misleading to give it a special name: One simply confines the use of the nonperturbative information to the light-light sector, where it is completely justified. In this case that means setting the scale with, for example, f_π or m_ρ. The straightforward Fermilab approach, with one loop perturbation theory [32], is then used for the heavy-lights.

It is also worth mentioning here the calculation of Ref. [19]. This is similar to Refs. [12,13] in that the B meson is reached by extrapolation from relatively small values of am_Q. However the improvement is done perturbatively, and the EKM normalization [26] (but not the shift $M_1 \to M_2$) is used for the central values. I believe that the errors quoted in Ref. [19] adequately account for all systematic effects (within the quenched approximation) of that calculation.

2.4. Unquenching f_B

Figure 6 shows the current status of f_B calculations with two flavors ($N_f = 2$) of dynam-

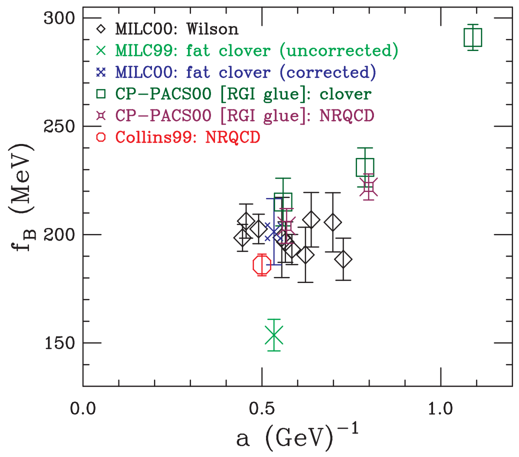

Figure 6.
World data for f_B with $N_f = 2$ *vs.* lattice spacing. Only statistical errors are shown. "MILC99" is Ref. [44]; see Table 1 for other references.

ical (sea) fermions. One improvement over last year is that the very low value from the MILC "fat-link" clover computation [44] has now been understood. From measurements of the quenched static quark potential, we have shown that fattening smooths out the potential well at short distances relative to the corresponding "thin-link" (standard) case. Indeed, for the fattening MILC used (10 iterations of APE smearing [45] with staple coefficient $\alpha = 0.45$), the potential at distance 1 (for $\beta = 5.85$) is increased by 45%. At distance $\sqrt{3}$, the increase is only to 8%. And by distance 3, it is under 1%. Such an effect is not surprising; the expected range for this amount of fattening is $\sim \sqrt{10 \times 4.5} \approx 2.1$ lattice spacings [46]. One expects that when the potential well at short distance becomes shallower, f_B decreases, since it is basically the wave function at the origin.

On quenched configurations at $\beta = 6.0$, $f_B^{\rm fat}/f_B^{\rm thin} = 0.76$ when "thin" is the NP-tad approach, and 0.73 when "thin" is NP-IOY. The extent of the effect was somewhat surprising to us, since the light hadron spectrum behaves quite well with fat-link clover fermions [47] and since the potential is only changed by an amount $> 10\%$ for distances < 0.16 fm. The difference between fat and thin may be exaggerated by the fact that in the fat case the one-loop renormalization of heavy-light Z_A has not been computed and the static-light Z_A [46,34] was used instead. Despite this, the ratio $f_B^{\rm fat}/f_B^{\rm thin}$ should be roughly the same at $N_f = 2, am = 0.01, \beta = 5.6$ (the parameters MILC used in the unquenched case) as at quenched $\beta = 6.0$ because the lattice spacings and couplings are quite close. Under this assumption, I plot $f_{B,N_f=2}^{\rm fat,corrected} = f_{B,N_f=2}^{\rm fat} \times f_{B,quench}^{\rm thin}/f_{B,quench}^{\rm fat}$ in Fig. 6, with thin=NP-tad.

The MILC result for $f_{B,N_f=2}$ in the continuum is obtained by extrapolating the Wilson data under two different assumptions that seem likely to bracket the real behavior: (1) constant behavior as $a \to 0$, and (2) a linear decrease from smallest current a values (≈ 0.45 GeV^{-1}) with a slope equal to that of the linear fit to the quenched data (see Fig. 4). The average of the two procedures is taken as the central value; the standard deviation, as the extrapolation error. The (preliminary) result shown in Fig. 1 and Table 1 then is: $f_B = 191(6)(^{+24}_{-18})(^{+11}_{-0})$ MeV, where the errors are respectively statistical, systematic within the $N_f = 2$ approximation, and an estimate of the error of neglecting the strange sea quark. Although the consistency of the corrected fat-link results with the Wilson data is comforting, the former is not included in MILC final results because there is no guarantee that the correction factor is the same in the quenched and full cases.

As in the quenched case, the preliminary CP-PACS results for f_B with $N_f = 2$ use the Iwasaki gauge action [22]. They find [17,18]: $f_{B,N_f=2} = 208(10)(11)$ MeV with the Fermilab formalism (heavy clover) and $204(8)(15)(+44)$ MeV with NRQCD; these results are shown in Fig. 1. The smallest-a points in Fig. 6 are taken as the "continuum" values, and the discretization error comes theoretical estimates: $\mathcal{O}(a^2\Lambda^2)$ and $\mathcal{O}(\alpha a\Lambda)$ in the heavy-clover case. However, my comment about discretization errors in the quenched CP-PACS data applies here too. Indeed, comparing Fig. 6 with Fig. 2, one sees that the lattice spacing dependence of the $N_f = 2$ data is if anything greater than in the quenched case. The theoretical error estimates do not account for the full amount of the variation with a. I therefore think the procedure for finding central values and errors is overly optimistic.

My own rough estimate from a $const + a^2$ fit to the CP-PACS heavy-clover data would be $f_{B,N_f=2} = 190(12)(26)$ MeV. The systematic error in my estimate is dominated by the discretization error, taken from the difference of the extrapolation and the smallest-a point. Although one can certainly argue with my extrapolation (there are for example $\alpha a \Lambda$ errors in addition to a^2 errors) I believe the systematic error of ~ 26 MeV is reasonable.

One difference between CP-PACS and MILC data shown in Fig. 6 is that the MILC data is "partially quenched" — the sea quark mass is held fixed while the valence mass is extrapolated to the physical u,d mass. The CP-PACS points are fully unquenched, with the valence and sea masses extrapolated together. However, MILC has repeated the analysis in a fully unquenched manner; the points are then still roughly constant in a, and the final result is changed only slightly. The difference is included in the MILC systematic error.

In the NRQCD $N_f = 2$ points in Fig. 1 [20,17], the systematic errors are quite large. The dominant source of these errors is the $\sim 25\%$ difference in the lattice scale from that set by light quark physics (m_ρ, say) and the 1P-1S Υ splitting. In previous, quenched simulations, it was generally assumed that the scale difference was due to quenching and would go away with dynamical quarks. Although the difference is in fact somewhat smaller for $N_f = 2$ than in the quenched approximation, it is still large; there is no indication that it would vanish in the physical case, *i.e.*, 3 light dynamical quarks. Perhaps the errors for heavy-heavy spectra in NRQCD are underestimated (see Sec. 5). I do not, however, have the data and calculations to subject this suspicion to a quantitative test.

Looking at Fig. 1 and recalling my remarks on discretization effects, one sees that the systematic errors on f_B in both the quenched and the 2-dynamical-flavor cases are still rather large overall. However, comparing quenched and $N_f = 2$ results by the same groups, for which many of the systematic effects cancel, one can say with some confidence that $f_{B,N_f=2}$ is about 10–15% larger than $f_{B,quench}$. The effect of unquenching is roughly the same for f_{B_s}; while the increase for f_D and f_{D_s} appears somewhat smaller, about 3–8% [44,16–18].

Taking into account all above remarks, I arrive at the following world averages:

$$
\begin{aligned}
&f_{B,quench} = 175(20)\text{ MeV}, \qquad &&f_B = 200(30)\text{ MeV},\\
&\left(\frac{f_{B_s}}{f_B}\right)_{quench} = 1.15(4), \qquad &&\frac{f_{B_s}}{f_B} = 1.16(4),\\
&f_{D_s} = 255(30)\text{ MeV}, && \qquad (8)
\end{aligned}
$$

where quantities without the "*quench*" subscript are supposed to be full QCD quantities, including an estimate of the effect (and error) that would result from including the strange sea quark. Starting with the quenched results and adjusting them by the expected increase in the full theory would result in similar unquenched values and errors.

3. B PARAMETERS

Figure 7 presents most of the recent world data on the B-$\bar{B}$ mixing parameter $\hat{B}_{B_d}$, where the "hat" indicates the renormalization group invariant quantity at NLO. Three types of calculations are shown: heavy-light with relativistic heavy quarks (here, clover quarks) extrapolated to the B meson mass [12,21,19], static-light [48,49], and NRQCD-light [50,51].

In the relativistic heavy-light case, the extrapolation to the B is motivated by the heavy quark effective theory. HQET implies that B_{P_d} is a power series (up to logs) in $1/M_{HL}$, where B_{P_d} is the (left-left) B parameter of a heavy-light meson P (mass M_{HL}) with a physical d quark and an arbitrary heavy quark. Both APE [12] and Lellouch and Lin [19] make linear fits in $1/M_{HL}$, resulting in the extrapolated points shown in Fig. 7. Although it must be admitted that the data of Ref. [19] fit the linear form quite well, I have concerns about trusting the HQET down to $M_{HL} \approx 1.2 GeV$ ($1/M_{HL} \approx 0.85$ GeV^{-1}), which is the lightest mass shown in Fig. 7. Indeed, since the relativistic points in Refs. [12,19] suffer from some of the same systematic effects discussed in Sec. 2.3 (*e.g.*, the choice of M_1 or M_2), the ex-

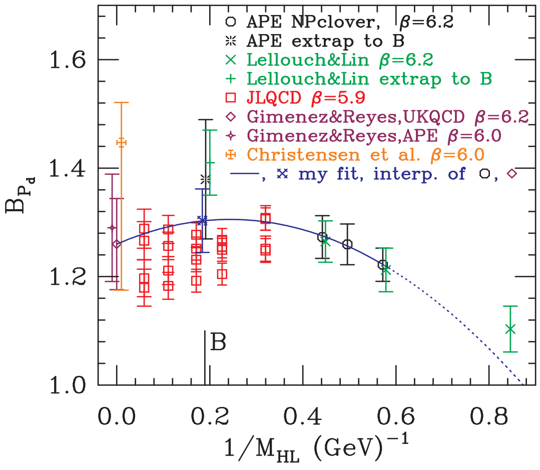

Figure 7.
Selection of recent data for quenched $\hat{B}_{P_d}$ *vs.* $1/M_{HL}$, where P is a general heavy-light pseudoscalar and M_{HL} is its mass. Only statistical errors are shown for heavy-light data and extrapolations [19,12,21]. Static-light points [48,49] include perturbative systematic errors. JLQCD [50,51] uses NRQCD-light; the 4 points at each mass represent different ways of treating the $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s p/m_Q)$ corrections.

tremely linear behavior may well be an accident. I therefore will omit the 0.85 GeV^{-1} point from my analysis.

References [12,19] do not take into account the static-light results (*e.g.*, Refs. [48,49]) in their extrapolations. APE justifies this by noting that the perturbative error on the static-light B parameter is large, while the heavy-light points in [12] use nonperturbative normalization. In my view, however, the $am_Q \sim 1$ systematic errors in the heavy-light case (sec. 2.3) are comparable to the perturbative errors of the static-light method. Therefore, I choose to make a quadratic fit (in $1/M_{HL}$) to the APE points as well as the Gimenez and Reyes/UKQCD static-light point. (The latter probably has the smallest systematic errors of the available static-light data, since it uses improved light quarks at the weakest coupling, $\beta = 6.2$.) My fit is shown in Fig. 7. At the B mass, it gives $\hat{B}_{B_d} = 1.30(6)$ (statistical error only). Note that the extension of the fit comes reasonably close to the 0.85 GeV^{-1} point; alternatively, including this point in the fit would produce only a 1% change in $\hat{B}_{B_d}$.

An important advance in the past year is the calculation [51] of the one-loop renormalization of the $\Delta B = 2$ 4-quark operators in the NRQCD-light case. Previously [52], the static-light renormalization constants had to be used instead, and the result for $\hat{B}_{B_d}$ was $\sim$20% low compared to the extrapolated values of Refs. [12,19] and still $\sim$15% below my interpolated value. (Note, however, that the perturbative error in Ref. [52] was estimated rather realistically as $\sim$ 10%.) The current perturbative calculation includes terms of $\mathcal{O}(\alpha_s/(am_Q))$, but not those of $\mathcal{O}(\alpha_s p/m_Q)$, where p is the typical quark 3-momentum in the B meson. The various squares at the same values of $1/M_{HL}$ in Fig. 7 represent an attempt to estimate the systematic errors of $\mathcal{O}(\alpha_s p/m_Q)$ as well as $\mathcal{O}(\alpha_s^2)$. The results are consistent with those of my interpolation. Unfortunately, I do not have the data to perform a renormalon shadow analysis on the NRQCD-light results, but my guess is that the effect is no larger than in the f_B case.

Taking into account my interpolation and the results of Refs. [50,51], as well as my estimate of the systematic errors in both, I quote $\hat{B}_{B_{d,quench}} = 1.30(12)$. It is not clear what quenching error should be assigned here. From quenched chiral perturbation theory (QChPT), Sharpe [53] estimates a 10% error. There are not many simulations that address this question. Gimenez and Reyes [54] find a $\sim$11% difference between quenched and unquenched results in the static-light case. However, the systematic effects in the two results are rather different, so it is difficult to draw a strong conclusion. MILC also has some quite preliminary results [55] in the static-light case, suggesting a smaller quenching error, but again the systematics in the comparison are not under good control. At this point therefore I quote the QChPT 10% as the quenching error on the "world average" $\hat{B}_{B_d}$:

$$\hat{B}_{B_d} = 1.30(12)(13); \;\; f_B\sqrt{\hat{B}_{B_d}} = 230(40)\ \mathrm{MeV}, \quad (9)$$

where the uncertainty on $f_B\sqrt{\hat{B}_{B_d}}$ is supposed to include all sources of error. Since QChPT for f_B [56] seems to overestimate somewhat the quenching errors, I regard the assumed 10% quenching effect on $\hat{B}_{B_d}$ as quite conservative.

For the ratio $\hat{B}_{B_s}/\hat{B}_{B_d}$, all groups get a number very close to 1. Furthermore, the experience with f_{B_s}/f_B, as well as a preliminary simulation [55], suggest that the quenching errors on the ratio are considerably smaller than the $\sim$5% inferred from QChPT [53]. My world averages are then:

$$\frac{\hat{B}_{B_s}}{\hat{B}_{B_d}} = 1.00(4); \quad \xi \equiv \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_{B_d}\sqrt{\hat{B}_{B_d}}} = 1.16(5), \quad (10)$$

where I have used Eq. (8). All sources of error, including quenching, are supposed to be represented in Eq. (10).

Before concluding this section, I wish to mention results for B_S, the B parameter for the scalar-pseudoscalar $\Delta B = 2$ operator, which is relevant for the Standard Model prediction of the width difference $\Delta\Gamma/\Gamma$ for B_s mesons. B_S has been computed with heavy quarks that are relativistic [57,58,12,21], nonrelativistic [50,51] and static [54]. The last reference includes a first look at unquenching effects. The results from all the groups appear consistent with $B_S^{\overline{MS}}(m_b) = 0.82(8)$, where I have defined $B_S^{\overline{MS}}$ as in [58], and have tried to include all sources of error. There are however large $1/m_Q$ corrections as well as disagreements in how to go from B_S to the width difference, so more work is required before we have a Standard Model prediction for $\Delta\Gamma_{B_s}/\Gamma_{B_s}$.

4. SEMILEPTONIC FORM FACTORS

New work this year on semileptonic decays has dealt almost exclusively with $B \to \pi\ell\nu$, so I will focus on that process only. The hadronic matrix element can be parameterized with form factors f_+ and f_0 defined by

$$\langle\pi(\vec{k})|V_\mu|B(\vec{p})\rangle = f_+(q^2)\Big[p_\mu + k_\mu - \frac{(M_B^2 - M_\pi^2)q_\mu}{q^2}\Big] + f_0(q^2)\frac{(M_B^2 - M_\pi^2)q_\mu}{q^2}, \quad (11)$$

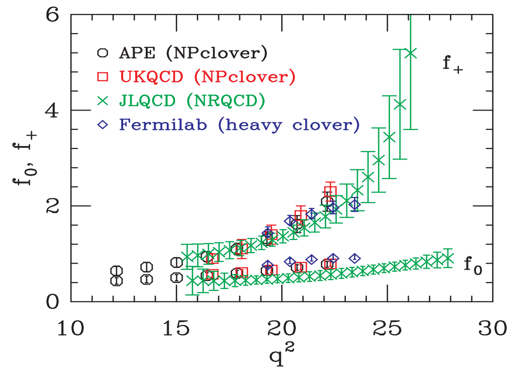

Figure 8.
Recent data for $B \to \pi\ell\nu$ formfactors f_+ and f_0. Only statistical errors are shown.

where V_μ is the vector current and $q \equiv p - k$ is the 4-momentum transfer to the leptons.

Figure 8 shows recent results for the form factors. Various approaches have been tried:[3] UKQCD [59,14] and APE [21] use relativistic quarks and extrapolate to the the B; JLQCD [60] treats the heavy quark with NRQCD; and the Fermilab group [61] uses their formalism to simulate with the heavy quark at the b mass.

In all the approaches, the first step in analysis of the raw lattice data for the form factors (or for $\langle\pi(\vec{k})|V_\mu|B(\vec{p})\rangle$ in the Fermilab case) is an interpolation or fit. For APE or UKQCD, this is a fit of f_+ and f_0 to expected phenomenological forms. APE uses the nice "B-K" form [63], which automatically incorporates the known scaling behavior in $1/M_{HL}$ at $q^2 = q^2_{max}$ (the end point) and at $q^2 = 0$, and enforces the kinematic constraint $f_+(0) = f_0(0)$. UKQCD employs more conventional forms such as pole/dipole, although fits to the B-K expressions do appear later in their analysis. In practice, what is important at this stage is that the fit smooths out data which is inherently rather noisy. For example, I refer the reader to Figure 2 in Ref. [14], which I believe is typical of the kind of data seen by all the groups.

[3] A form factor study using coarse anisotropic lattices [62] is not far enough along yet to be included in this comparison.

One must thus keep in mind, when looking at plots like Fig. 8, that the apparent smoothness of the data is a result of a procedure that starts with an "averaging" process over a range in q^2 and therefore produces highly correlated points.

The next step is a chiral extrapolation. In the heavy meson rest frame, $q^2 = M_{HL}^2 + M_\pi^2 - 2M_{HL}\sqrt{M_\pi^2 + \vec{k}^2}$. This means that an extrapolation of form factors at fixed, small $\vec{k}$ should contain a term linear in M_π, through the implicit q^2 dependence, in addition to the usual powers of M_π^2 [64]. Alternatively, one can perform the extrapolation at fixed q^2 [59] or fixed $v \cdot k$ [21,60] (v is the heavy meson 4-velocity), or simply extrapolate the B-K parameters (or similar ones) rather than the form factors themselves [21]. Yet another choice is to avoid the small $\vec{k}$ region altogether (the decay rate is after all small there) and make a standard chiral extrapolation at fixed $\vec{k}$ [61].

UKQCD and APE then need to extrapolate in $1/M_{HL}$ to the B mesons. UKQCD extrapolates f_+ and f_0 at fixed $v \cdot k$, assuming that q^2 is close enough to q^2_{max} that the HQET forms valid near the end point may be used. APE extrapolates the B-K parameters themselves (and also repeats the UKQCD-type analysis as a check). The two extrapolations are consistent within errors (see Fig. 8), suggesting that the HQET endpoint forms are indeed appropriate. Of course, just as for f_B and B_B, either extrapolation introduces two sources of systematic error: (1) uncertainty in what order polynomial (in $1/M_{HL}$) one should use, and (2) magnification of small $\mathcal{O}(a^2 M_{HL}^2)$ errors. Both groups do an adequate job of estimating (1). However, I think that error (2) could use more study along the lines of Sec. 2.3.

JLQCD and the Fermilab group work in the region of the B mesons, so extrapolation in the heavy mass is not required. Although their results in Fig. 8 are similar, there may be some quantitative disagreement, especially in the normalization of f_0 and in the q^2 dependences of both form factors. I have some concern that the Fermilab data for f_+ does not appear to show the expected pole-like rise toward the end point seen by the other groups. (Remember that each group's points are highly correlated, so that the shape of the q^2 dependence is probably highly significant.) Of course, for APE and UKQCD, this behavior is put in *ab initio* by the fit to the phenomenological form. JLQCD, on the other hand, does not appear to be forcing this behavior by their analysis procedure, so the fact that it does come out seems encouraging.

Since the Fermilab and JLQCD (NRQCD) approaches are fundamentally quite similar in the B meson region, it is not easy to guess the source of the differences. One possibility is discretization error. The Fermilab group (alone among the four) extrapolates to the continuum from a range of a ($\beta = 5.7$, 5.9, 6.1), while JLQCD works only at $\beta = 5.9$. However, the a-dependence seen by the Fermilab group is mild, and it seems unlikely to me that this is the explanation.

Perhaps a more likely culprit is the current renormalization. JLQCD is working to one-loop order in perturbation theory. The Fermilab group does much of the renormalization nonperturbatively (by comparing to the known normalization of the diagonal vector current), but the remainder is still put in at tadpole-improved tree level for the data in Fig. 8. They will be including the complete one-loop calculation shortly, and they assure me that it makes little numerical difference. It will, however, be interesting to investigate the size of the renormalon shadows for both the JLQCD and the Fermilab computations.

Another issue that has bedeviled lattice $B \to \pi\ell\nu$ calculations for several years is the soft-pion relation (SPR) at the end point, $f_0(q^2_{max}) = f_B/f_\pi$ [65,66]. Some early calculations found violations by as much as a factor of 2 [67], and although the situation has improved somewhat, the problem has not gone away.

At first sight, it seems surprising that the endpoint region should give trouble. After all, lattice calculations are easiest at the end point where no 3-momentum insertion is required ($\vec{k} = 0$ in the B rest frame). However, the SPR is only guaranteed to be valid in the limit $M_\pi \to 0$. As discussed above, the extrapolation in M_π when $\vec{k}$ is fixed and small is tricky because q^2 changes and has a term linear in $M_\pi \propto \sqrt{m_{quark}}$. APE and

UKQCD get consistency with the SPR using chiral extrapolations that avoid sitting at the end point (instead fixing q^2 or $v \cdot k$). The end point is then reapproached after extrapolation to the physical q^2_{max} and the B meson mass. But, after all the extrapolations, the systematic errors are rather large, so the agreement with the SPR is not very convincing. Indeed, in the case of APE, the SPR is used as a criterion to choose whether one should do linear or quadratic fits in $1/M_{HL}$, so the agreement cannot then be seen as a success of the computation.

JLQCD, on the other hand, studies the SPR in detail by sitting at the end point with high statistics (2150 configurations). Figure 9 shows the chiral extrapolation of $f_0(q^2_{max})$. Note that the fit (solid line) which takes into account the implicit linear dependence on M_π comes closer to f_B/f_π than the purely quadratic fit in M_π but still misses by a wide margin. This illustrates how hard it will be to verify the soft-pion relation directly: the linear dependence on M_π causes a sharp rise, but only at very small mass where there is no data. This behavior is expected from the effects of q^2 poles in the form factor in the unphysical region just beyond the end point. A global fit to the functional dependence on $v \cdot k$ and M_π expected from the HQET-suggested form factors of Burdman *et al.* [66], does somewhat better, but still does not give convincing agreement with f_B/f_π. The Fermilab group has independently decided to use the form factors of Ref. [66] to study the end-point behavior, and it will be very interesting to see how their results compare to those of JLQCD.

Of course, one can always take the attitude that the SPR is irrelevant to phenomenology because there is no rate near the end point and, besides, it is f_+, not f_0, that determines the rate into $\pi\mu\nu$ or $\pi e\nu$. However, a credible demonstration of the SPR would be very helpful in convincing the non-lattice community that we have the form factor computation under control.

5. SPECTRA

Although I do not have space for a detailed review of recent spectral calculations for hadrons with one or more heavy quarks (Refs. [68–80]), I will make some remarks about a few features I find interesting and/or confusing:

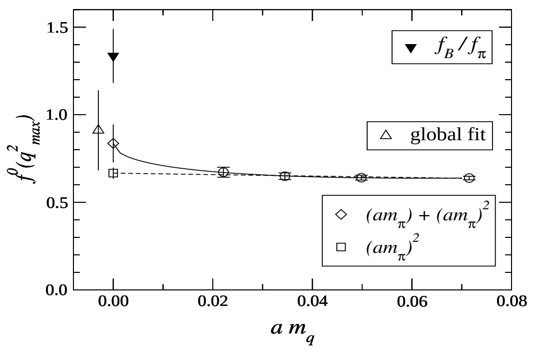

Figure 9.
Extrapolation of f_0 at the end point to the chiral limit by JLQCD [60]. The open triangle comes from a global fit to the form in Ref. [66].

(1) It seems clear that the computed quenched hyperfine splittings for heavy-light mesons are considerably smaller than the experimental values. For example, $M_{B^*_s} - M_{B_s}$ in the quenched approximation appears to be about half its physical value [68,71,72], although some older calculations *e.g.*, Ref [81], found smaller ($\sim 20\%$) discrepancies. For the D_s system, the effect is also present at about the 20% level in most [71,82,81] calculations (with one recent exception[70]). The sign and, very roughly, the magnitude of the observed discrepancy is expected from arguments about the effects of quark loops on the short distance potential. However, I know of no direct evidence from simulations that quenching is the culprit.

(2) There is a disagreement between Refs. [72] and [68] in the computed P-wave splittings for B mesons ($M_{B^*_2} - M_{B^*_0}$ or $M_{B^*_0} - M_B$, or the corresponding quantities for B_s). The disagreement is considerably larger than the systematic errors quoted by both groups and deserves further study. My guess is that the problem lies in excited state contamination: These mass differences are quite noisy, and it is seems very difficult to identify a "true plateau" in the effective mass plots.

(3) For NRQCD computations of splittings in charmonium, Stewart and Koniuk [75] confirm previous evidence [83] that the results are very sensitive to the order (in the velocity v) to which the Hamiltonian is corrected and how the tadpole improvement is done (plaquette or Landau link). Indeed, including the $\mathcal{O}(v^6)$ terms, with the renormalization done at tadpole-improved tree level, moves the computed hyperfine splitting significantly *further* from experiment. I believe this is a "smoking gun" for renormalon shadow effects: At tadpole-improved tree level the presence of higher dimensional operators has no effect on the coefficients of lower dimensional operators — there is no subtraction of power law divergences. Just as for f_B, putting in the higher dimensional operators without the subtraction is then worse than leaving out those operators entirely. Unlike the f_B case, however, the effect here appears to be numerically dramatic.

(4) In the $b\bar{b}$ system, NRQCD is clearly better behaved than in $c\bar{c}$, since the relativistic corrections are smaller. It is therefore possible to find convincing evidence for quenching effects [77,73] if one is careful to keep other systematic effects (lattice spacing, order in v, type of tadpole improvement) fixed when comparing quenched and unquenched simulations. However, even for $b\bar{b}$, I believe it is important to reduce renormalon shadows by doing at least one-loop renormalization of the Hamiltonian before one can make reliable comparisons with experiment for most quantities.

(5) Following pioneering work by Klassen [84], there have been several recent simulations of charmonium on quenched "anisotropic relativistic" lattices [76–78]. The anisotropy ξ is defined by $\xi \equiv a_s/a_t$, where a_s and a_t are the spatial and temporal lattice spacings, respectively. Choosing $\xi > 1$ (typically $2 \leq \xi \leq 5$) allows one to keep $a_t m_Q \ll 1$ and avoid large mass dependence in the improvement coefficients. Effectively, the heavy quark is relativistic.[4] Although the improvement coefficients through $\mathcal{O}(a)$ can in principle be evaluated nonperturbatively, in practice the spatial and temporal clover coefficients are

[4] To see that $a_s M \sim 1$ does not introduce uncontrolled mass dependence requires [85] an analysis along the lines of [26].

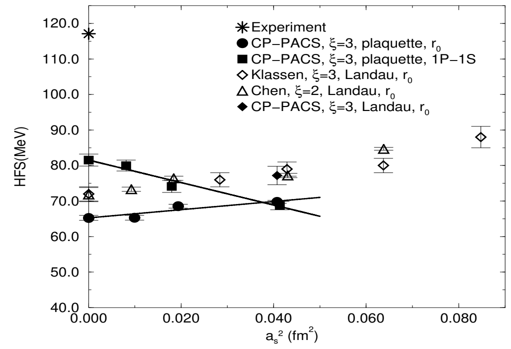

Figure 10.
Hyperfine splitting in charmonium *vs.* a_s^2 for relativistic anisotropic lattices. Data is from Refs. [84,76,77]; figure is taken from [77].

determined at tadpole-improved tree level, either by the Landau link or the plaquette.

Figure 10 shows the continuum extrapolation for the charmonium hyperfine splitting for various choices of anisotropy, tadpole factor, and scale determination. The good news is that the results from this method seem quite precise compared to previous relativistic calculations on isotropic lattices [86] and do not have the sensitive dependence on the tadpole factor seen [75,83] in NRQCD. The bad news is that there appears to be some remaining dependence on the tadpole factor in the continuum limit (compare the open triangles and filled circles).[5] The reason for the discrepancy is not yet clear. Additional data at larger a_s with the plaquette tadpole may be useful in sorting out what is happening.

(6) The anisotropic relativistic approach has also been quite helpful in getting good signals for quarkonium hybrids [78]. As in glueball calculations [87], anisotropy gives more usable time slices before the signal is lost in noise. Another recent hybrid calculation [69] also employs anisotropic lattices, but in the NRQCD context.

(7) Kronfeld [79] has recently argued that a

[5] Differences in the continuum value due to different scale choices (r_0 or the 1P-1S splitting) are presumably explained by quenching and/or phenomenological uncertainties in r_0 (see Sec. 2).

heavy quark on the lattice can be represented by a *continuum* HQET where the dependence on am_Q is completely absorbed into the short-distance coefficients of the heavy quark operators. The operator matrix elements then only have mild lattice spacing dependence though the light degrees of freedom. This nice insight is then applied [80] to the heavy-light meson spectrum on the lattice in an attempt to compute the HQET parameters $\bar{\Lambda}$ and λ_1. ($\bar{\Lambda}$ is the meson binding energy in the static limit; $-\lambda_1/2m_Q$, the heavy quark kinetic energy.)

I have some concerns about Ref. [80]. One could find the continuum HQET by a two-step process: (a) for fixed a in the lattice theory, take m_Q large and arrive at a *lattice* HQET, and (b) order by order in perturbation theory, replace the lattice regularized HQET operators by their continuum counterparts. Although this not the procedure employed in [80], I believe the approaches should be equivalent. If I am right, then there are effective power law divergences and renormalon shadow effects[6] in the coefficients of the continuum operators (resulting from step (b)). Therefore, a one-loop calculation may not be sufficiently accurate to extract $\bar{\Lambda}$ and λ_1. Indeed, parameterizing the discretization errors on $\bar{\Lambda}$ as $\bar{\Lambda}(a) = \bar{\Lambda}(0)(1 - a\mathcal{M})$, the slope $\mathcal{M}$ has a value of ≈ 0.65 GeV. Yet, if the lattice spacing dependence is coming only from the light degrees of freedom, which are at least tree-level improved through $\mathcal{O}(a)$, one expects $\mathcal{M} = \mathcal{O}(\alpha_s \Lambda_{QCD})$, an order of magnitude smaller than what is found. This may be a renormalon shadow effect.

6. CONCLUDING REMARK

I have not made a unitarity triangle analysis with my world average lattice results. However, Fig. 11 shows the result of an analysis [88] that employs lattice values similar to those quoted here.[7] Clearly, our knowledge of the triangle is getting quite precise. Over the last dozen years, the allowed region for the vertex has shrunk in area by almost a factor of 20, as is shown dramatically in Fig. 4 of Ref. [89]. We can be proud of the fact that lattice computations have played an important role in this accomplishment.

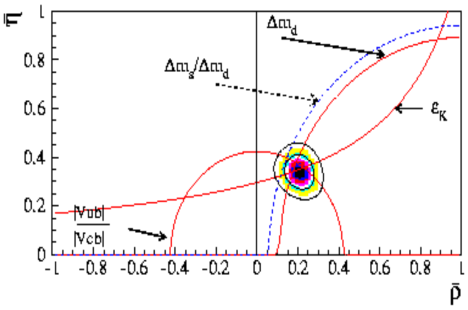

Figure 11.
Allowed region for the vertex of the unitary triangle (CKM parameters $\bar{\rho}$ and $\bar{\eta}$) from Ref. [88].

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[6] In the Fermilab formalism, there are no true power law divergences (aside from the standard additive shift in the mass) as $a \to 0$ for fixed m_Q because the relativistic theory is regained. The coefficients (*e.g.*, d_1) vanish fast enough to cancel the power divergence in the higher dimensional operators. However, as discussed for f_B in Sec. 2.1, the power divergences of the operators themselves imply that the renormalon shadow issue is still present.

[7] The main difference is that [88] uses the value $f_B\sqrt{\hat{B}_{B_d}} = 220(25)(20)$ MeV rather than my 230(40) MeV.

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