A calculation of the Lepage-Mackenzie scale for the lattice axial and vector currents

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We calculate the perturbative scales ($q^\star$) for the axial and vector currents for the Wilson action, with and without tadpole improvement, using Lepage and Mackenzie's formalism. The scale for the pseudoscalar density (times the mass) is computed as well. Contrary to naive expectation, tadpole improvement reduces $q^\star$ by only a small amount for the operators we consider. We also discuss the use of a nonperturbative coupling to calculate the perturbative scale.

I. INTRODUCTION

The calculation of weak matrix elements, such as the f_B decay constant, is one of the most important applications of lattice QCD, because it is this nonperturbative physics that obscures possible physics beyond the standard model. Although lattice QCD offers a nonperturbative method of calculating weak matrix elements from first principles, in practice a perturbative calculation is also required to extract the continuum number. Lattice perturbation theory is much harder than continuum perturbation theory, because there is less symmetry on the lattice that can be used to simplify expressions. This means that lattice perturbative quantities are typically only known to one loop in perturbation theory. It seems important to extract as much information from the one-loop calculation as possible.

In the past, one-loop lattice perturbation theory was considered unreliable, because for many quantities the one-loop corrections are large if one uses the bare lattice coupling constant, and the agreement with results that could be calculated numerically was the very poor. Lepage and Mackenzie [1] then showed that the use of a more "physical" coupling, calculated at a judiciously chosen scale (generically referred to as $q^\star$) and the resummation of gluon-tadpole contributions to the perturbative expansion improve the situation for many quantities.

In this paper we calculate the scale $q^\star$ using the Lepage and Mackenzie prescription for the local axial and vector currents (as well as the mass times the pseudoscalar density), with and without tadpole improvement. Some of our results were also obtained in Ref. [2]; however that paper did not study tadpole-improved perturbation theory. In addition, the current work shows how to deal with certain ambiguous (though numerically small) contributions to the calculation in Ref. [2].

There are a number of nonperturbative renormalization techniques that have been used in numerical computations [3–5]. These techniques are very interesting, but they have not as yet been widely used in large scale simulations. The calculation of the nonperturbative renormalization constants required in the calculation of f_B in the simulations by the MILC collaboration [6], for example, would require, at the least, a substantial investment in computer time. There are also conceptual problems in generalizing existing nonperturbative methods to the case of large quark mass. Furthermore, it is an important consistency check on the numerical calculation of renormalization factors, that they should agree with the weak-coupling results at small enough coupling, because there are potential systematic errors, such as lattice-spacing artifacts, in the numerical results. This motivates trying to extract as much information as possible from lattice perturbation theory calculations.

The outline of this paper is as follows: in section 2 we calculate the renormalization constants from Ward identities, while in section 3, they are calculated by matching on-shell matrix elements to the continuum. In section 4, these renormalization constants are tadpole improved, and in section 5, we calculate the scales $q^\star$ for each of them with and without tadpole improvement. In section 6, we discuss the stability of our results under changes in the scale-setting prescription. We end by providing some insight into the fact why tadpole improvement does not lower the scale $q^\star$ for these quantities as much as one might naively expect.

II. Z_A, Z_{mP} AND Z_V FROM WARD-IDENTITIES

First, we will derive the renormalization factors from the Ward identities [7,8]. We consider the following Ward identity (WI) for the nonsinglet flavor axial current $A_\mu = A^a_\mu T^a$ (T^a are the hermitian, traceless flavor generators; we work in euclidean space):

$$\begin{aligned}\langle\sum_\mu \partial_\mu A^a_\mu(x) q(z)\overline{q}(y)\rangle &= 2m\langle P^a(x)q(z)\overline{q}(y)\rangle \\ &\quad -\langle q(z)\overline{q}(y)\rangle T^a\gamma_5\delta(x-y) - T^a\gamma_5\langle q(z)\overline{q}(y)\rangle\delta(x-z)\ ,\end{aligned} \tag{2.1}$$

with

$$A^a_\mu(x) = \overline{q}(x)\gamma_\mu\gamma_5 T^a q(x)\ , \tag{2.2}$$

$$P(x) = \overline{q}(x)\gamma_5 T^a q(x)\ . \tag{2.3}$$

This WI holds for any regulator which respects the (nonsinglet) axial flavor symmetry, such as naive dimensional regularization (NDR), in which γ_5 anticommutes with γ_μ for all $\mu = 1,\ldots,d$. Since A^a_μ is the only dimension-three axial vector operator that can be built out of quark and gluon fields, it does not mix with other operators. Taking $m = 0$ in eq. (2.1), we see that A^a_μ does not require renormalization (the RHS of eq. (2.1) is finite after renormalizing the quark fields $q(z)$ and $\overline{q}(y)$). Thus, if Z_A is defined to be the renormalization constant for A^a_μ which preserves the WI (2.1), we have $Z_A = 1$ in any scheme, such as NDR, which preserves the axial flavor symmetry. The operators mP^a also do not mix with any other operators, and therefore Z_{mP} (defined to be the renormalization constant of the operator mP^a) may be determined by considering the case $m \neq 0$ after Z_A has been defined. In NDR, we have $Z_{mP} = 1$.

Wilson fermions break the axial flavor symmetry, and no conserved axial currents exist, so that the WI introduced above does not hold for any axial current on the lattice. Here, we are interested in the local (*i.e.*, single-site) axial current and pseudo-scalar density as defined in eqs. (2.2,2.3), and we would like to find the multiplicative renormalization constants Z_A and Z_{mP} such that eq. (2.1) holds in the scaling region:

$$\begin{aligned}Z_A\langle\sum_\mu \partial_\mu A^a_\mu(x) q(z)\overline{q}(y)\rangle &= 2Z_{mP}(M-M_c)\langle P^a(x)q(z)\overline{q}(y)\rangle \\ &\quad -\langle q(z)\overline{q}(y)\rangle T^a\gamma_5\delta(x-y) - T^a\gamma_5\langle q(z)\overline{q}(y)\rangle\delta(x-z)\ .\end{aligned} \tag{2.4}$$

Here $M = m + 4r$ (r is the Wilson parameter and m is the bare mass) is the coefficient of the single-site term in the Wilson–Dirac lagrangian, and M_c is its critical value. On the lattice, we may write

$$\begin{aligned}\langle\sum_\mu \Delta_\mu A^a_\mu(x) q(z)\overline{q}(y)\rangle &= 2(M-M_c)\langle P^a(x)q(z)\overline{q}(y)\rangle \\ &\quad -\langle q(z)\overline{q}(y)\rangle T^a\gamma_5\delta(x-y) - T^a\gamma_5\langle q(z)\overline{q}(y)\rangle\delta(x-z) \\ &\quad +\langle E^a(x)q(z)\overline{q}(y)\rangle\ ,\end{aligned} \tag{2.5}$$

where Δ_μ is the backward difference operator $\Delta_\mu f(x) = f(x) - f(x-\mu)$, and E^a is the "evanescent" operator

$$\begin{aligned}E^a(x) &= \sum_\mu \Delta_\mu A^a_\mu(x) - 2(M-M_c)P^a(x) \\ &\quad +\overline{q}(x)\gamma_5 T^a(D+M)q(x) + \overline{q}(x)(\overleftarrow{D}+M)\gamma_5 T^a q(x)\ ,\end{aligned} \tag{2.6}$$

in which

$$Dq(x) = \frac{1}{2}\sum_\mu\left[(\gamma_\mu - r)U_\mu(x)q(x+\mu) - (\gamma_\mu + r)U^\dagger_\mu(x-\mu)q(x-\mu)\right] \tag{2.7}$$

($D + M$ is the Wilson–Dirac operator).

Going to momentum space by multiplying eq. (2.5) by $\sum_{xyz}\exp(-ikx - ipz + ip'y)$, and amputating the external quark lines, we find, at one-loop order in g,

$$\langle E^a(k)q(p)\overline{q}(p')\rangle^{\rm one-loop}_{\rm amp} = C_F g^2\delta(p+k-p')T^a I(p,p')\ , \tag{2.8}$$

with $C_F = \frac{N^2-1}{2N}$ the quadratic Casimir for $SU(N)$, and

$$\begin{aligned}
I(p,p') = -\sum_\nu (1-e^{i(p-p')_\nu}) \sum_\mu \int_\ell D(\ell) \times \qquad\qquad & (2.9)\\
\Big(\Gamma^-_\mu S(p+\ell)\gamma_\nu\gamma_5 S(p'+\ell)\Gamma^-_\mu \ e^{i(p+p'+\ell)_\mu} & \\
+ \Gamma^-_\mu S(p+\ell)\gamma_\nu\gamma_5 S(p'+\ell)\Gamma^+_\mu \ e^{i(p-p')_\mu} & \\
+ \Gamma^+_\mu S(p+\ell)\gamma_\nu\gamma_5 S(p'+\ell)\Gamma^-_\mu \ e^{-i(p-p')_\mu} & \\
+ \Gamma^+_\mu S(p+\ell)\gamma_\nu\gamma_5 S(p'+\ell)\Gamma^+_\mu \ e^{-i(p+p'+\ell)_\mu} \Big) & \\
+2(M-M_c)\sum_\mu \int_\ell D(\ell) \Big(\Gamma^-_\mu S(p+\ell)\gamma_5 S(p'+\ell)\Gamma^-_\mu \ e^{i(p+p'+\ell)_\mu} & \\
+ \Gamma^-_\mu S(p+\ell)\gamma_5 S(p'+\ell)\Gamma^+_\mu \ e^{i(p-p')_\mu} & \\
+ \Gamma^+_\mu S(p+\ell)\gamma_5 S(p'+\ell)\Gamma^-_\mu \ e^{-i(p-p')_\mu} & \\
+ \Gamma^+_\mu S(p+\ell)\gamma_5 S(p'+\ell)\Gamma^+_\mu \ e^{-i(p+p'+\ell)_\mu} \Big) & \\
-\Sigma(p,m)\gamma_5 - \gamma_5\Sigma(p',m) \ , &
\end{aligned}$$

$$\begin{aligned}
\Sigma(p,m) = \sum_\mu \int_\ell D(\ell) \Big(-\Gamma^-_\mu S(p+\ell)\Gamma^-_\mu \ e^{i(2p+\ell)_\mu} - \Gamma^+_\mu S(p+\ell)\Gamma^+_\mu \ e^{-i(2p+\ell)_\mu} & \\
- \Gamma^-_\mu S(p+\ell)\Gamma^+_\mu - \Gamma^+_\mu S(p+\ell)\Gamma^-_\mu \qquad\qquad & (2.10)\\
+ \frac{i}{2}\sin p_\mu \gamma_\mu - \frac{r}{2}\cos p_\mu \Big) \ . &
\end{aligned}$$

Here $\Sigma(p,m)$ is the one-loop fermion self-energy (defined to be equal to the sum of diagrams, with no overall minus sign), and $S(p) = \left[i\sum_\mu \gamma_\mu \sin p_\mu + m + r\sum_\mu (1-\cos p_\mu)\right]^{-1}$ is the tree-level fermion propagator. The gluon propagator in the Feynman gauge is $\delta_{\mu\nu}D(p)$ with $D^{-1}(p) = 4\sum_\mu \sin^2(p_\mu/2)$. The matrices $\Gamma^\pm_\mu$ are defined by

$$\Gamma^\pm_\mu = \frac{1}{2}(\gamma_\mu \pm r) \ . \qquad (2.11)$$

Furthermore, we abbreviated $\int_\ell \equiv \frac{1}{(2\pi)^4}\int_{|\ell_\mu|\le\pi} d^4\ell$.

We now observe that $I(0,0) = -2\gamma_5\Sigma(0,0)$ provides the counterterm to remove the $1/a$ divergence in the contact terms in eq. (2.5) (after amputation of external fermion lines). A straightforward calculation then shows that the remainder, $I(p,p') - I(0,0)$, is finite, and we can take the continuum limit. We expand to linear order in p, p' and $m = M - M_c + O(g^2)$:

$$C_F\,(I(p,p') - I(0,0)) = c_A \ i(\not p' - \not p)\gamma_5 - 2c_{mP} \ m\gamma_5 \ , \qquad (2.12)$$

from which we then obtain the renormalization constants

$$Z_A = 1 - g^2 c_A \ , \qquad Z_{mP} = 1 - g^2 c_{mP} \ , \qquad (2.13)$$

with, for $N=3$ and $r=1$,

$$c_A = 0.133373(2)\ , \qquad c_{mP} = 0.081419(2)\ . \tag{2.14}$$

At this point, we pause to comment on this derivation. Generally, in calculating the continuum limit of lattice integrals such as $I(p,p')$, special care is required near the origin in momentum space. Near $\ell = 0$ in eq. (2.9), one may approximate the integrand by its covariant form. In this case, with the routing of the external momenta through the fermion line, it turns out that the covariant form of the integrand vanishes identically (as explained below), and therefore eq. (2.12) is obtained by a straightforward expansion in p, p' and m. This would not be true had we chosen the routing of the external momenta through the gluon line. This will turn out to be relevant for the definition of $q^\star$, and will be discussed further below.

A completely analogous analysis may be performed in order to find the renormalization constant Z_V of the local vector current

$$V^a_\mu(x) = \overline{q}(x)\gamma_\mu T^a q(x)\ . \tag{2.15}$$

For Wilson fermions, this current is not conserved, and we find

$$Z_V = 1 - g^2 c_V\ , \qquad c_V = 0.174083(3)\ . \tag{2.16}$$

Equations (2.14,2.16) are in agreement with previous computations [9–11].

III. Z_A, Z_{mP} AND Z_V BY ON-SHELL MATCHING TO THE CONTINUUM

A common approach in the literature is to compute the renormalization constants by matching an on-shell lattice perturbative computation to the continuum, where the renormalization constants are known. We present this approach here because the value of $q^\star$ depends on the specific form of the integrands. It is therefore important to check that a separate derivation of the integrals will produce a consistent result for $q^\star$.

For definiteness, we focus first on the nonsinglet axial current, A_μ. The derivations for the vector (V_μ) and pseudoscalar (mP) cases are similar, but there are a few important differences, which we note explicitly. Because A_μ has no anomalous dimension or mixings, we may write its one-loop matrix element between on-shell quark states i and f in regularization scheme S as

$$\langle f \mid (A^a_\mu)^S \mid i\rangle = \gamma_\mu\gamma_5 T^a (1 + g^2 \mathcal{C}^S_A(m,\lambda))\ , \tag{3.1}$$

where m is the quark mass, λ is a gluon mass that has been inserted to regularize the infrared divergences, and spinors on the external lines are implicit.

We then define a on-shell renormalized axial current $(A^a_\mu)^{\rm shell}$

$$(A^a_\mu)^{\rm shell} \equiv \tilde{Z}^S_A(m,\lambda)(A^a_\mu)^S \tag{3.2}$$

$$\tilde{Z}^S_A(m,\lambda) = 1 - g^2\mathcal{C}^S_A(m,\lambda) + \dots\ . \tag{3.3}$$

By definition, the on-shell quark matrix element of $(A^a_\mu)^{\rm shell}$ is the same in any scheme and equal to $\gamma_\mu\gamma_5 T^a$, as long as m and λ are the same as in eq. (3.3). Because there is no operator mixing, this then implies the scheme independence of all renormalized Green's functions with a $(A^a_\mu)^{\rm shell}$ insertion. However, we may also define a renormalized current $(A^a_\mu)^{\rm WI}$ using the Ward Identity, eq. (2.4):

$$(A^a_\mu)^{\rm WI} \equiv Z^S_A\ (A^a_\mu)^S\ , \tag{3.4}$$

where the superscript S specifies the scheme to which eq. (2.4) is applied. Z^S_A is UV finite; it is free of infrared divergences since it is defined off-shell. By dimensional analysis, Z^S_A is therefore also independent of the quark mass m. Since renormalized Green's functions with a $(A^a_\mu)^{\rm WI}$ insertion are also scheme independent, we have, for any two schemes S and S',

$$\frac{Z^S_A}{\tilde{Z}^S_A(m,\lambda)} = \frac{Z^{S'}_A}{\tilde{Z}^{S'}_A(m,\lambda)} \tag{3.5}$$

We now specialize to the case $S =$ lattice and $S' =$ NDR. As long as the regulator does not violate the symmetry, the Lie algebra of conserved charges fixes the on-shell matrix elements of the corresponding currents, guaranteeing that a conserved nonabelian flavor current is not renormalized. Therefore

$$\tilde{Z}_A^{\rm NDR}(m=0,\lambda) = 1 \tag{3.6}$$

since the nonsinglet axial current is conserved when $m = 0$. Similarly,

$$\tilde{Z}_V^{\rm NDR}(m,\lambda) = 1 \tag{3.7}$$

for *all* m, λ. A simple computation shows, however, that $\tilde{Z}_A^{\rm NDR}(m,\lambda) \neq 1$ for general m, λ, (including the limit $\lambda \to 0$, m fixed). Note that the presence of the gluon mass in eqs. (3.6, 3.7) does not violate the flavor symmetries, although it would violate gauged symmetries.

It is useful to see explicitly how eqs. (3.7) and (3.6) arise at one-loop. At this order, the computations are identical to those in QED, and amount the statement that $Z_1 = Z_2$ [12]. If one routes the external momentum p through the fermion line, then the cancellation of the wave-function and vertex renormalizations which results in eq. (3.7) can easily be shown from the (Euclidean) identity

$$\frac{\partial}{\partial p^\mu}\frac{1}{(i\not p + m)} = \frac{1}{(i\not p + m)}(-i\gamma_\mu)\frac{1}{(i\not p + m)} \tag{3.8}$$

Equation (3.8), coupled with the on-shell definition of the wave-function renormalization,

$$Z_2 = 1 + \left.\frac{p^\mu}{m}\frac{\partial\Sigma}{\partial p^\mu}\right|_{i\not p = -m} , \tag{3.9}$$

immediately gives $Z_1 = Z_2$. With the chosen momentum routing, the equality holds at the level of the Feynman integrands. On the other hand, if the external momentum is routed through the gluon line, there is no obvious relation between the integrands for Z_1 and Z_2. The equality then appears only after the momentum and Feynman parameter integrals are performed and requires a regulator which permits shifts of the loop momentum without the generation of boundary terms. This is the reason that the "covariant part" of the lattice integral in Section II vanishes only when the external momentum is routed through the fermion line. Although complete lattice integrals are invariant under shifts, the "covariant part" is not because it is the integral of a continuum-like integrand over a sphere of radius δ (the "inner region" of Ref. [13]) or of radius π (the integration region chosen for the continuum-like integrals added and subtracted in Ref. [14]). The boundary term generated by a shift in the loop momentum thus does not vanish in the continuum limit for a linearly divergent integral such as the self-energy.

For mP, one can relate the mass derivative of the self-energy to the vertex function (at least at one loop), and again show at the level of the Feynman integrands (with the proper momentum routing) that

$$\tilde{Z}_{mP}^{\rm NDR}(m=0,\lambda) = 1 \ . \tag{3.10}$$

Returning to the discussion of $Z_A^{\rm lattice}$, we combine eqs. (3.5) and (3.6), and use $Z_A^{\rm NDR} = 1$ (since NDR preserves the chiral symmetry), giving

$$Z_A^{\rm lattice} = \tilde{Z}_A^{\rm lattice}(0,\lambda) \ . \tag{3.11}$$

The corresponding relation holds for $Z_V^{\rm lattice}$ and $Z_{mP}^{\rm lattice}$. Note that all direct reference to the continuum has been eliminated. Since from now on we talk only about lattice quantities we can drop the qualifier "lattice" from eq. (3.11) and return to the notation of Section II. We remark that the renormalization factors were computed in Ref. [11] using the equivalent of eq. (3.11).

The computation of the right hand side of eq. (3.11) is fairly easy, since it can be done in the massless limit. We need to compute the self-energy graphs (the continuum-like graph and the lattice tadpole) and the vertex correction graph (for each current). To avoid computing any covariant parts, we route the external momentum through the fermion line. This routing will also be very convenient in the computation of $q^\star$.

We write the lattice self-energy as

$$\Sigma(p) = \frac{1}{a}\Sigma_0(p^2,m^2,\lambda^2) + i\not p\,\Sigma_1(p^2,m^2,\lambda^2) + m\Sigma_2(p^2,m^2,\lambda^2) \ . \tag{3.12}$$

Only Σ_1 is needed for wave function renormalization at $m = 0$; Σ_2 is needed for Z_{mP}. Computing the self-energy graphs with the methods of Ref. [14], we have

$$\Sigma_{1,2}(0,0,\lambda^2) = g^2 C_F \int_\ell I_{1,2} \ . \tag{3.13}$$

Taking Wilson $r=1$, the integrands $I_{1,2}$ are:

$$\begin{aligned}
I_1 &= \frac{1}{8\Delta_1} + \frac{1}{4\Delta_1\Delta_2}\left(-2 + \frac{11\Delta_1}{2} - 2\Delta_1^2 - \frac{\Delta_4}{2} + \frac{1}{\Delta_2}\left(\Delta_4 - \Delta_5 + \Delta_1\Delta_4 + 4\Delta_1\Delta_5 - 2\Delta_1^2\Delta_4 - 2\Delta_{13}\right)\right) + \\
&\quad + \frac{\theta(\pi^2-\ell^2)}{\ell^4} + covariant\ part, \\
I_2 &= -\frac{4\Delta_1(\Delta_1-2)+\Delta_4}{\Delta_2^2} + \frac{\Delta_1 - 2}{2\Delta_1\Delta_2} + \frac{4\theta(\pi^2-\ell^2)}{\ell^4} + covariant\ part,
\end{aligned} \tag{3.14}$$

where $\Delta_1 \dots \Delta_5$ are defined in Refs. [14,15] (for $r=1$):

$$\begin{aligned}
\Delta_1 &= \sum_\alpha \sin^2(\frac{\ell_\alpha}{2}) \ , \\
\Delta_4 &= \sum_\alpha \sin^2(\ell_\alpha) \ , \\
\Delta_5 &= \sum_\alpha \sin^2(\frac{\ell_\alpha}{2}) \sin^2(\ell_\alpha) \ , \\
\Delta_2 &= \Delta_4 + 4\Delta_1^2 \ ,
\end{aligned} \tag{3.15}$$

and we introduce

$$\Delta_{13} \equiv \sum_\alpha \sin^4(\frac{\ell_\alpha}{2}) \sin^2(\ell_\alpha) \ . \tag{3.16}$$

The "covariant parts" in eq. (3.14) do not need to be specified further, since they will cancel in the final answers, as explained above. The terms proportional to $1/\ell^4$, which come from the expansion in powers of a of added and subtracted covariant integrals, are written explicitly here, since they are needed to cancel the logarithmic divergences in the rest of the integrals. However, these terms will also cancel in the final results.

In Ref. [14], the external momentum in the self-energy computation was chosen to flow through the gluon line. (This saved some effort in the expansion in powers of a, since the fermion propagator is more complicated than the gluon one.) For our current purposes, we therefore have performed the calculation of I_1 from scratch. In the case of I_2, however, one may set $p=0$ from the beginning, so that the routing of the external momentum is irrelevant. Then I_2 may be taken immediately from Ref. [14].

For the vertex diagrams we may also set $p=0$ *ab initio* and take the results from Ref. [14]. We have

$$\Lambda_{\gamma_\mu\gamma_5,\gamma_\mu,\gamma_5} = (\gamma_\mu\gamma_5, \gamma_\mu, \gamma_5) g^2 C_F \int_\ell \mathcal{J}_{\gamma_\mu\gamma_5,\gamma_\mu,\gamma_5} \ , \tag{3.17}$$

where

$$\begin{aligned}
\mathcal{J}_{\gamma_\mu\gamma_5} &= 4\mathcal{K}_2' + 2\mathcal{K}_1 + \mathcal{K}_0 \ , \\
\mathcal{J}_{\gamma_\mu} &= 4\mathcal{K}_2' - 2\mathcal{K}_1 + \mathcal{K}_0 \ , \\
\mathcal{J}_{\gamma_5} &= 16\mathcal{K}_2' - 4\mathcal{K}_1 + \mathcal{K}_0 \ ,
\end{aligned} \tag{3.18}$$

with

$$\begin{aligned}
\mathcal{K}_2' &= \frac{1}{48\Delta_2^2}\left(\frac{\Delta_5}{\Delta_1} - \Delta_4\right) + \frac{\Delta_4}{16\Delta_1\Delta_2^2} - \frac{\theta(\pi^2-\ell^2)}{4\ell^4} + covariant\ part, \\
\mathcal{K}_1 &= \frac{1}{16\Delta_2^2}\left(\Delta_4 + 4\Delta_1(\Delta_1 - 4)\right) \ , \\
\mathcal{K}_0 &= \frac{1}{12\Delta_2^2}\left(\Delta_4 - 4\frac{\Delta_5}{\Delta_1} + 6\Delta_4 + 12\Delta_1^2\right) \ .
\end{aligned} \tag{3.19}$$

If we now write the Z factors in terms of the constants c_A, c_V and c_{mP} as in eqs. (2.13,2.16), we have

$$c_A = C_F \int_\ell (I_1 + \mathcal{J}_{\gamma_\mu\gamma_5}) \ ,$$

$$c_V = C_F \int_\ell (I_1 + \mathcal{J}_{\gamma_\mu}) ,$$
$$c_{mP} = C_F \int_\ell (I_2 + \mathcal{J}_{\gamma_5}) . \tag{3.20}$$

Explicit calculation shows that the covariant parts of $I_{1,2}$ and $\mathcal{J}_{\gamma_\mu\gamma_5,\gamma_\mu,\gamma_5}$ cancel in eq. (3.20), as expected from the general arguments above. Numerical integration of eq. (3.20) then reproduces eqs. (2.14,2.16).

IV. TADPOLE IMPROVEMENT

The idea behind tadpole improvement [1] is based on the observation that, for the relatively large values of the lattice coupling constant g used in current lattice QCD computations, the expectation value of the link variable, $u_0 \equiv \langle U_{\mu x}\rangle$ (defined, *e.g.*, in Landau gauge), is not very close to one. If one would calculate, for instance, the quark propagator in a mean-field approximation by replacing $U_{\mu x} \to u_0$ in the quark action, one would find

$$\langle q(x)\overline{q}(y)\rangle_{\rm MF} = u_0^{-1}\langle q(x)\overline{q}(y)\rangle_{\rm free}(M \to u_0^{-1}M) , \tag{4.1}$$

where the notation $M \to u_0^{-1}M$ indicates that we replace M by $u_0^{-1}M$ in the expression for the free Wilson propagator.

This implies that, in order to make contact with the continuum, the quark field has to be renormalized by a factor $\sqrt{u_0}$. For local bilinears like $A_\mu^a(x)$, $P^a(x)$ and $V_\mu^a(x)$, this results in a factor u_0:

$$A_\mu^a(x)_{\rm cont} = u_0 A_\mu^a(x)_{\rm lattice} , \tag{4.2}$$

and similarly for the other operators. From the mean-field result eq. (4.1) for the quark propagator, we find that $M_c = 4ru_0$ and $m_{\rm MF} = u_0^{-1}(M - M_c)$. (In terms of the hopping parameter $\kappa = 1/(2M)$, we get $\kappa_c = 1/(8ru_0)$.) Therefore, the operator $(M - M_c)P^a(x)$ does not renormalize at all.

The factor u_0 takes into account the gauge-field tadpole diagrams contributing to the renormalization constants Z_A and Z_V. Of course, these are not the only contributions. We may now use standard perturbation theory to improve on our mean-field estimates for the renormalization constants, but we should be careful to take out those contributions that have already been absorbed into u_0 [16]. It follows that

$$A_\mu^a(x)_{\rm cont} = \frac{u_0}{u_0^{\rm PT}} Z_A A_\mu^a(x)_{\rm lattice} , \tag{4.3}$$

where $u_0^{\rm PT}$ is the mean link $\langle U_{\mu x}\rangle$ calculated in perturbation theory, u_0 is the nonperturbatively (*i.e.*, numerically) computed value (using the same definition!), and Z_A is the usual renormalization factor, calculated to one loop in eq. (2.13). This leads to the definition of the tadpole-improved perturbative renormalization factor

$$\hat{Z}_A = Z_A/u_0^{\rm PT} , \tag{4.4}$$

with a similar definition of $\hat{Z}_V$.

The mean-field link, u_0, can be defined in different ways, and the idea of tadpole improvement is useful to the extent that the various definitions agree. Recently, Lepage [17] and Trottier [18] have advocated the use of the Landau mean link as the nonperturbative u_0 factor. In Landau gauge, to one loop, we have (for $N = 3$)

$$u_0^{\rm PT} = 1 - 0.077466(1)g^2 \qquad \text{(Landau gauge)}. \tag{4.5}$$

We may also calculate the mean link from M_c:

$$u_0^{\rm PT} = 1 - 0.108570(2)g^2 \qquad \text{(from } M_c\text{)}. \tag{4.6}$$

We may now write

$$\hat{Z}_i = 1 - c_i g^2 , \qquad i = A, V , \tag{4.7}$$

and tabulate the numerical values for the constants c_i obtained from eqs. (4.4–4.7). For $u_0^{\rm PT}$ calculated in Landau gauge or from M_c, we report the values in table I.

V. RESULTS FOR THE LEPAGE-MACKENZIE SCALE $q^\star$

The prescription for obtaining the Lepage-Mackenzie scale is to calculate $q^\star$ from

$$\log((q^\star)^2) \equiv \frac{\int d^4q f(q) \log(q^2)}{\int d^4q f(q))} \ , \tag{5.1}$$

if the one-loop integral for a particular renormalization constant has the form $\int d^4q f(q)$, and where q is the momentum flowing through the gluon line (see below). The condition that the momentum flow through the gluon line is critical both for the understanding and the correct application of the Lepage and Mackenzie scale-setting procedure.

The Lepage and Mackenzie procedure is based on an older technique for setting the scale in continuum perturbative expressions due to Brodsky, Lepage, and Mackenzie (BLM) [19]. In the BLM procedure, the scale is chosen to remove the leading-order coefficient of the number of fermion flavors from the perturbative expansion. Physically this absorbs a class of graphs from the fermion contribution to the vacuum polarization into the running coupling. This boils down to the requirement that q in eq. (5.1) is the momentum flowing through the gluon propagator for self-energy and vertex-correction diagrams.

The original BLM scale setting procedure is more general than the Lepage and Mackenzie prescription (eq. (5.1)), because it can cope with perturbative expressions of operators with nonzero anomalous dimensions. The connection between the Lepage and Mackenzie scale setting procedure and the BLM procedure has been discussed by Kronfeld [20].

We will now calculate $q^\star$, both with and without tadpole improvement, using eq. (5.1). Let us, for definiteness, consider the case of Z_A. The relevant integral is given in eq. (3.20), with I_1 and $\mathcal{J}_{\gamma_\mu\gamma_5}$ given by eqs. (3.14,3.18,3.19). Since the loop momentum q has been chosen from the beginning to be the gluon momentum (*i.e.*, the external momentum is routed through the fermion line), we just need to recompute eq. (3.20) with $\log(q^2)$ inserted.

In table II, we give the values of $q^\star$ for the three Z_i, and for the three cases of no improvement, Landau-gauge tadpole improvement, and M_c tadpole improvement. For the case of Z_{mP}, we did not include $q^\star$ values from tadpole improvement, since the operator $mP^A(x)$ is not affected by it, as explained in the previous section.

Note that if we had chosen the loop momentum to be the fermion momentum, the insertion of $\log(q^2)$ into the integrands would in general give a different result. While we may always shift the loop momentum in the original expressions for the diagrams, this is not true after we have inserted $\log(q^2)$. Indeed, in the case where q is the fermion momentum, the integral with $\log(q^2)$ inserted is not even convergent. The covariant parts in that case generate integrals roughly of the type

$$\int d^4q \frac{a^2p^2}{(q^2+a^2p^2)^3} \ , \tag{5.2}$$

which approaches a constant as $a \to 0$ but diverges logarithmically if $\log(q^2)$ is inserted into the integrand before $a \to 0$ is taken. This tells us that this procedure of determining the typical scale $q^\star$ is consistent at one loop only when this scale is the typical momentum of the gluon propagator. This is in accordance with the intuitive arguments underlying the BLM scheme.

In previous computations of $q^\star$ in the static-light case [21] and in the Wilson or clover case [2], the loop momentum was not chosen to be the gluon momentum. The straightforward insertion of $\log(q^2)$ into the integrals was therefore not possible, because the "constant terms" (the $a \to 0$ limit of integrals like eq. (5.2)) would give divergent results. These terms then had to be treated in an ad-hoc manner: Hernández and Hill [21] replaced the integrals with constants over the Brillouin zone; Crisafulli *et al.* shifted and redefined the integration variables until an integral was found which remained convergent after insertion of $\log(q^2)$. Luckily, however, these terms contribute only a small amount to the final results for $q^\star$. Therefore the ambiguities in previous computations make only a small numerical difference. The results in Ref. [2] for the $q^\star$ of Z_A or Z_V differ by about 1% from the current, unambiguous answers. In making this comparison, we computed the integrals in Ref. [2] for ourselves (after correcting two typographical errors in I_Σ in eq. (139)), since the quoted answers ($q^\star = 2.4$ for Z_V and $q^\star = 2.6$ for Z_A without tadpole improvement) had too few significant figures to see the difference clearly.

Since the integrals which define the c_i (eqs. (2.12,2.16) or (3.20)) are finite and represent cutoff effects, one would anticipate $q^\star$ for these quantities to be of order π/a. In the tadpole-improved case, where one hopes the most severe cutoff effects are taken out of the perturbative renormalization parts, one might expect the values of $q^\star$ to be much lower [1]. Table II shows that, while $q^\star$ is indeed reduced for tadpole-improved quantities, this reduction is rather minimal, and tadpole-improved $q^\star$'s are still substantially larger than $1/a$.

VI. SCALE SETTING USING A NONPERTURBATIVE EXPRESSION FOR THE COUPLING

The prescription for computing $q^\star$ given in eq. (5.1) can be motivated starting from the assumption that a natural value for $q^\star$ would be obtained from [1]

$$\alpha(q^\star)\int d^4 q f(q) = \int d^4 q f(q)\alpha(q) \ . \tag{6.1}$$

The idea here is to use a coupling that depends on the internal momentum in the diagram to weight the integrand, rather than multiplying the diagram with a coupling at a constant scale. Unfortunately the right hand side of eq. (6.1) is singular if the perturbative renormalization group improved evolution equation is used for the coupling, because of the Landau singularity. Lepage and Mackenzie avoided this problem, by using the one loop perturbative evolution equation for the coupling (which is free of the Landau singularity) in eq. (6.1) to derive eq. (5.1).

As a consistency check on the scale obtained from eq. (5.1), it is instructive to try to use a nonperturbative definition of the coupling in eq. (6.1). The Landau singularity in the renormalization group evolution equation for the coupling occurs when the coupling is large, and the perturbative derivation of the evolution equation breaks down. Neubert [22] developed a formalism based on eq. (6.1) to calculate the scale for a number of continuum perturbative expressions. He found that his more general formalism produced well behaved perturbative expression for some quantities that were previously thought to be unreliable because of their low BLM scale (the continuum analog of using eq. (5.1)). This motivates studying the use of eq. (6.1) for lattice perturbation theory.

As the running coupling of QCD has been obtained nonperturbatively, from a number of lattice QCD simulations [23–25], it seems natural to try a coupling measured from a lattice QCD simulation in eq. (6.1) to estimate the scale for lattice perturbative expressions. A useful expression for a nonperturbative QCD coupling for scale setting was suggested by Klassen [26]. This QCD coupling is defined by fitting the static potential between two quarks in momentum space to an ansatz for the evolution of the coupling that has no Landau pole and reduces to the perturbative expressions in the weak coupling limit. Klassen used eq. (6.1) with his nonperturbative coupling to estimate the scale for the plaquette, which was consistent with the result from Lepage and Mackenzie's formalism.

Unfortunately, combining Klassen's nonperturbative definition of the coupling with the integrands for Z_A in eq. (6.1) produces a divergent integral. The problem is that the integrands for Z_A behave like $O(\frac{1}{q^2})$ for small momentum q. Klassen's coupling also behaves like $O(\frac{1}{q^2})$, thus producing a divergent result. The physical reason for the low momentum limit of Klassen's coupling is the assumption that the static quark potential at large distances is linear in the quark separation. However, it is expected that at a certain separation between the quarks, string breaking will occur, which will cause the potential to flatten out. The crossover from a linearly rising potential to a flat potential has never been seen in lattice gauge theory simulations (see Refs. [27,28] for recent discussions). ¿From lattice gauge theory, it is thus currently impossible to extract a coupling in momentum space, for use in eq. (6.1), that has the correct low momentum limit.

There is some evidence from experiment that the low energy limit of the coupling tends to a finite limit in the small momentum limit. In the theoretical analysis of e^+e^- event shapes, Dokshitzer and Webber [29] introduced the low momentum mean of the coupling. Their fit to experimental data yielded

$$\frac{1}{2\ GeV}\int_0^{2\ GeV} \alpha(\mu^2) d\mu = 0.52 \pm 0.04 \ . \tag{6.2}$$

If a low momentum coupling with the limit of eq. (6.2) exists, then it would have a finite limit with the integrands for Z_A.

Inspired by the work that obtained eq. (6.2), Shirkov and Solovtsov [30] derived an expression for the coupling without a Landau pole:

$$\alpha_{ss}(k^2) = \frac{1}{\beta_0}\left(\frac{1}{\log(k^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - k^2}\right) \ . \tag{6.3}$$

Equation (6.3) is based on a dispersive theory and the one-loop RG evolution equation. The choice of scheme is reflected in the Λ parameter. This definition of the coupling, as well as its connection to renormalons, is reviewed in Refs. [31,32]. The analytic coupling in eq. (6.3) has been used to study the scale dependence in the continuum [33].

Although eq. (6.3) could just be substituted into the right-hand side of eq. (6.1) to provide an answer for the one-loop perturbative contribution, it is convenient to calculate a $q^\star$ to compare with the scale obtained from the Lepage and Mackenzie scale. We define a $q^\star$ from:

$$\frac{1}{\log\left((q^\star)^2/\lambda^2\right)} + \frac{1}{1-(q^\star)^2/\lambda^2} = \frac{\int d^4q f(q)\left(1/\log(\frac{q^2}{\lambda^2}) + \frac{1}{1-q^2/\lambda^2}\right)}{\int d^4q f(q)} \ , \tag{6.4}$$

where q and $\lambda \equiv a\Lambda$ are dimensionless. Using $\lambda = 0.14$ ($\lambda = 0.0587$), taken from Ref. [26] at $\beta = 6.0$ ($\beta = 6.8$), we obtain $q^\star = 2.32$ ($q^\star = 2.33$) for Z_A, and $q^\star = 2.10$ ($q^\star = 2.10$) for the tadpole improved (using M_c) Z_A. The two scales are both close to ones obtained from the Lepage and Mackenzie scheme. Again we see that tadpole improving the perturbative factor only causes a small reduction in the scale. Although eq. (6.4) is less convenient to use with simulation data, because of the dependence on the scale through λ, this formalism does provide a check on the Lepage and Mackenzie scale-setting procedure.

VII. VISUALIZATION OF THE INTEGRANDS

To try to understand why the estimate of the scale for tadpole improved perturbative factors was not close to the naive expectation $\frac{1}{a}$, and to investigate the effects of having different weighting functions, we wanted to "visualize" the lattice integrands. Consider a lattice approximation to a continuum integral

$$\int_0^{2\pi} f(q) d^4q \sim \frac{1}{L^4} \sum_{x=0}^{L-1} \sum_{y=0}^{L-1} \sum_{z=0}^{L-1} \sum_{t=0}^{L-1} f(2\pi x/L, 2\pi y/L, 2\pi z/L, 2\pi t/L) \tag{7.1}$$

$$= \frac{1}{L^4} \sum_{q_E} f(q_E) n(q_E) \ , \tag{7.2}$$

where q_E is a single member of the equivalence class under hypercubic transformations. The function $n(q_E)$ gives the number of momenta in each equivalence class. This technique is the basis of an efficient method to integrate lattice Feynman diagrams [34].

In figure 1, we plot the $f(q_E)$ function for the Z_A integrand with and without tadpole improvement. The magnitude of the tadpole-improved integrand is much reduced over the original integrand. However, if we choose the normalization of the tadpole-improved integrand to equal the not improved integrand at a specific momentum, then the two figures agree qualitatively. Because the Lepage-Mackenzie scale does not depend on the overall normalization of the integrands, it is perhaps not surprising that the scales for the tadpole and nontadpole Z_A are so close.

VIII. CONCLUSION

Our main results for the perturbative scales are contained in table II. Tadpole improvement does not significantly reduce $q^\star$ for the operators considered here. In particular, the scales for the tadpole-improved operators are higher than the scale $q^\star = 1/a$ suggested by the intuitive idea that tadpole improvement reduces the effect of the momentum near the cut off [1].

The closeness of the scale for the tadpole-improved Z_A to the scale for the tadpole-improved Z_A in the static limit [21] ($aq^\star = 2.18$) seems tp indicate that the mass dependence of $q^\star$ is weak. This suggests that using a mass-independent $q^\star$ with the results of the mass-dependent perturbative calculation of Z_A by Kuramashi [35] is a good approximation. This was the procedure adopted in the calculation of f_B by the MILC collaboration [6].

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	no improvement	Landau gauge	through M_c
c_A	0.133373(2)	0.055907(1)	0.024803(1)
c_V	0.174083(3)	0.096617(2)	0.065512(1)
c_{mP}	0.081419(2)	–	–

TABLE I. One-loop constants for tadpole-improved renormalization constants.

quantity	no improvement	Landau gauge	through M_c
Z_A	2.533	2.235	2.316
Z_V	2.370	2.090	2.051
Z_{mP}	1.905		

TABLE II. The scale $q^\star$ (in units of $1/a$) for various quantities and improvement schemes. The errors are less than 1 in the last place.

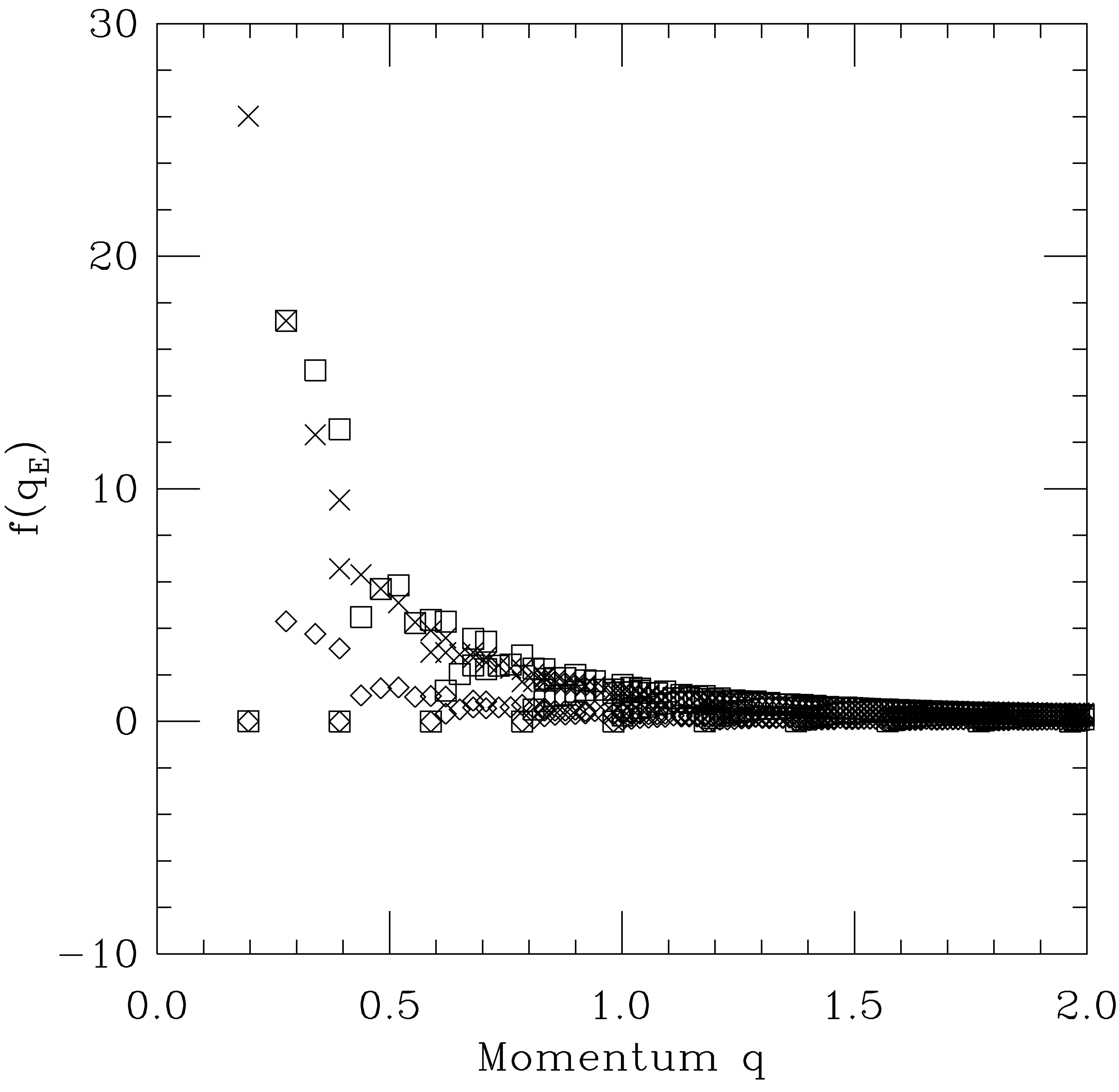

FIG. 1. Plots of the integrand for Z_A with (diamonds) and without (crosses) tadpole improvement (through M_c). The squares are the tadpole improved integrand normalized to agree with the nontadpole improved numbers at a specific point. The lattice volume was 32^4. For clarity only momentum with magnitude less than two are displayed.