Partially Quenched Gauge Theories and an Application to Staggered Fermions

Submitted to Physical Review D

Claude W. Bernard and Maarten F.L. Golterman

Department of Physics CB1105
Washington University
St. Louis, MO 63130-4899, USA

Abstract

We extend our lagrangian technique for chiral perturbation theory for quenched QCD to include theories in which only some of the quarks are quenched. We discuss the relationship between the partially quenched theory and a theory in which only the unquenched quarks are present. We also investigate the peculiar infrared divergences associated with the η' in the quenched approximation, and find the conditions under which such divergences can appear in a partially quenched theory. We then apply our results to staggered fermion QCD in which the square root of the fermion determinant is taken, using the observation that this should correspond to a theory with four quarks, two of which are quenched.

e-mail: cb@lump.wustl.edu, maarten@wuphys.wustl.edu

Wash. U. HEP/93-32 May, 1993

1. Introduction

There has been a growing interest in chiral perturbation theory (ChPT) for the quenched approximation of QCD [1-5]. The motivation for this is the fact that at present the quenched approximation is indispensable for the numerical study of QCD. Since ChPT is used in order to analyze and extrapolate numerical data, it is necessary to adapt it to the quenched approximation. To this end, we have developed a systematic, lagrangian technique that can be used straightforwardly to calculate quenched correlation functions within ChPT [3]. Quenched ChPT has been applied to calculate several important quantities, such as masses and decay constants [1-5].

However, aside from these practical results, it turns out that the special role of the η' in the quenched approximation leads to a very singular infrared behavior of quenched ChPT, which suggests that quenched QCD does not have a chiral limit [3-5]. Unlike the case in the full theory, the mass of the η' cannot be taken to infinity so as to leave us with an effective action which describes only the pseudo-Goldstone mesons. Instead, a double pole term proportional to the singlet part of the η' mass squared shows up in the η' twopoint function and causes the peculiar infrared behavior.

In this paper, we wish to extend our investigations to the case of partially quenched theories. Partially quenched theories are theories in which not all fermions are quenched; only for some of the fermions present in the theory will the determinant in the functional integral be replaced by 1. (We assume throughout a bilinear fermion action with only flavor diagonal terms.) The same method that we developed for studying ChPT for a completely quenched theory can also be applied in this case. Our motivation for considering partially quenched theories is threefold:

First, one may learn more about the peculiar infrared behavior by considering what happens when only part of the fermion content of a theory is quenched. In particular, if different fermion mass scales are present, one might ask how the infrared behavior depends on whether all or only some of the fermions with a common mass are quenched. Also, it is interesting to know what happens in the unquenched sector of the theory: is a theory with n fermions, out of which k are quenched, the same as an unquenched theory with just $n-k$ fermions? In the first three sections of this paper we address these questions.

Second, partially quenched theories arise naturally in the description of simulations in which the valence quark masses are not chosen equal to the sea-quark masses. This is a not uncommon numerical technique which, for example, allows one to use Wilson valence quarks and staggered sea-quarks. One would like to have a chiral theory for such simulations.

A third motivation comes from staggered fermions themselves. It is well known that lattice QCD with staggered fermions describes QCD with four flavors of quarks in the continuum limit. In order to use these fermions for simulations of QCD with only two flavors, a trick which has been used is to take the square root of the fermion determinant, thereby effectively reducing the number of flavors which appear in virtual quark loops from four to two.

This approach seems justified in weak coupling perturbation theory, but of course the question is whether it is really a legitimate technique. Certainly, taking the square root is not equivalent to formulating a two flavor theory through a functional integral with a local lagrangian. In the continuum limit, however, taking the square root of a four flavor fermion determinant (with at least pairwise degenerate quark masses) is exactly equivalent to quenching two out of four flavors, and our results about partially quenched ChPT with $n = 4$ and $k = 2$ should apply. This then allows us to test the idea of taking the square root within the context of ChPT: if this trick is legitimate, and does indeed lead to a two flavor theory, that should be reflected in ChPT. It means that partially quenched ChPT should reproduce the results of unquenched ChPT as long as we allow only unquenched quarks on the external lines of correlation functions (including operators which excite bound states of unquenched quarks).

Using results obtained in the first part of the paper, we show that this is indeed the case. This is a nontrivial test of the trick of taking the square root and complements the argument based on weak coupling perturbation theory, since ChPT addresses a different regime of QCD. As a corollary, we present a simple way in which $\eta'_{SU(2)}$ correlation functions can be computed in numerical simulations.

2. Theorems

In this section we state three theorems about partially quenched gauge theories and then give the physical arguments which underlie these results. We leave to sections 3 and 4 the detailed calculations in quenched chiral perturbation theory [3] which illustrate the theorems and various corollaries. For the first two theorems, the physical arguments actually constitute proofs, and the calculations of section 4 serve as explicit examples. For the third theorem, however, we give only an argument here only for a special case; the full proof will have to wait until section 3.

We first need to establish our notation. Consider a QCD-like theory with n flavors of quarks, q_i. The quark masses m_i $(i = 1, \ldots n)$ are completely arbitrary. We then partially quench this theory by adding k flavors $(0 \leq k \leq n)$ of pseudoquarks (bosonic quarks), $\tilde{q}_j$, as in ref. [3]. Note that the limiting cases of complete quenching or no quenching are allowed. The masses of the pseudoquarks are fixed to be equal to the masses of the first k real quarks, *i.e.*, m_j $(j = 1, \ldots k)$. In other words the first k quarks are "quenched," and the remaining $n - k$ quarks are "unquenched." We call this theory the "$SU(n|k)$ theory." We do not consider the more general, but probably physically uninteresting, case where the masses of the pseudoquarks are arbitrary — in that case there can be virtual pseudoquark loops that do not cancel completely against real quark loops. Note that, if there are degeneracies between some unquenched and some quenched quarks, *i.e.*, the quarks of some mass scale are only partially quenched, one is free to choose which of the quarks shall be the considered the "unquenched" ones. In other words, if among a degenerate quarks, b $(b < a)$ are quenched, one may arbitrarily choose which b quarks are to have indices j, with $1 \leq j \leq k$, and which $b - a$ quarks are to have indices r, with $k + 1 \leq r \leq n - k$.

A normal, completely unquenched theory with n quarks will be denoted as an "$SU(n)$ theory;" it is obviously the same as the $SU(n|0)$ theory.

The full chiral symmetry of the $SU(n|k)$ theory is the semi-direct product[1] $[SU(n|k)_L \otimes SU(n|k)_R] \circledS U(1)$, where the additional $U(1)$ present in $U(n|k)_L \otimes U(n|k)_R$ is broken by the anomaly.

The special case where all the m_i $(i = 1, \ldots n)$ masses are equal is called the "degenerate $SU(n|k)$ theory." In section 4, we also examine another special case where the number of quarks and pseudoquarks are even $(n \to 2n,\ k \to 2k)$ and the masses just take on two values: $m_1 = m_3 = m_5 = \cdots = m_{2n-1} \equiv m_u$, and $m_2 = m_4 = m_6 = \cdots = m_{2n} \equiv m_d$. We call this the "doublet $SU(2n|2k)$" theory.

We define the "super-η'" of the $SU(n|k)$ theory by the interpolating field

$$\Phi_0 = c\Big(\sum_{i=1}^{n} \overline{q}_i \gamma_5 q_i + \sum_{j=1}^{k} \overline{\tilde{q}}_j \gamma_5 \tilde{q}_j\Big) \ . \tag{2.1}$$

[1] The product is obviously direct when $k = 0$.

The normalization factor c was taken to be $1/\sqrt{n+k}$ ($i.e.$, $1/\sqrt{6}$ for $n=k=3$) in ref. [3], but has been left arbitrary here, since different normalizations will often be useful. The reader may be confused by the fact the super-η' does not appear explicitly as the *difference* of the quark η' and the pseudoquark η', as it does in the corresponding chiral theory, but rather as their sum. The reason is that the mesonic fields of a chiral theory directly correspond not to $\overline{q}\gamma_5 q$, but to $tr(q\overline{q}\gamma_5)$, since it is the *first* index of the meson matrix Σ which is the quark index. Taking into account the opposite statistics of quarks and pseudoquarks, the relative minus sign between quark and pseudoquark η' would reappear in eq. (2.1) when written in the latter form.

We can now state the three theorems about partially quenched theories:

I. In the subsector where all valence quarks are unquenched ($i.e.$, where all valence quarks are of type r, where $k+1 \leq r \leq n$), the $SU(n|k)$ theory is completely equivalent to a normal, completely unquenched $SU(n-k)$ theory.

II. The super-η' (with normalization $c = 1/\sqrt{n-k}$ in eq. (2.1)) is equivalent to the η' constructed in the unquenched sector of the $SU(n|k)$ theory, and is therefore, by I, equivalent to the $SU(n-k)$ η'. "Equivalent" here means that Green's functions constructed from an arbitrary number of super-η' fields and unquenched quarks, will be equal to the corresponding Green's functions with the super-η' replaced by the η' of the unquenched sector of the $SU(n|k)$ theory (or, what is the same, by the $SU(n-k)$ η'). Green's functions which involve the super-η' and arbitrary combinations of quenched quarks or pseudoquarks are not allowed — there is nothing for them to correspond to in the $SU(n-k)$ theory.

III. Quenched infrared divergences[3-5], coming from a double pole in the η' propagator and associated with some quark mass of mass m_j, will arise if and only if the scale m_j is fully quenched, $i.e.$, if there is a pseudoquark of mass m_j for every quark of mass m_j. In other words, these unphysical divergences arise if and only if $1 \leq j \leq k$ (so that this quark is quenched) and $m_j \neq m_r$, for all r with $k+1 \leq r \leq n$ (so that there are no unquenched quarks of the same mass).

Theorem I is easily established by a simple argument. Since by supposition all the valence quarks are unquenched, the only way the amplitudes could "know" about the quenched quarks and pseudoquarks is through virtual loops. But the pseudoquarks have been chosen to cancel the quenched quarks exactly in virtual loops, so only the unquenched quarks can appear anywhere in a diagram.

[Instructions for do-it-yourself graphics:

For Fig. 1(a), draw an oval (racetrack) with its long axis horizontal. Put an "x" on each short end (*i.e.*, the extreme right and left edges).

For Fig. 1b), draw an two ovals as in (a) and put them end to end. Put an "x" on the left edge of of the left oval and on the right edge of the right oval.

Good going, you did it!]

(a) (b)

Figure 1

Quark flow diagrams for the η' propagator; (a) is the "straight-through" diagram, and (b) is the "two-hairpin" diagram. Arbitrary numbers of gluon corrections and virtual quark loops (if the theory is not fully quenched) are implicit.

Theorem II also relies on the cancellation between quenched quarks and the pseudoquarks, but this time in valence lines. There are two quark flow diagrams that contribute to the η' propagator: the "straight-through" diagram (Fig. 1a) and the "two-hairpin" diagram (Fig. 1b). Note that we only specify the valence quark lines in these diagrams; in general (for $n \neq k$) there will be additional virtual quark loops. By the definition of the super-η' and the opposite statistics of quarks and pseudoquarks, the pseudoquarks will cancel the quenched quarks both in the straight-through and the two-hairpin diagrams (in the latter case, the cancellation takes place separately in each hairpin). Only the unquenched quarks survive in each diagram, and they of course will also be the only survivors in any virtual quark loops. Thus the contraction of any two super-η' fields is the same as the contraction of two $SU(n-k)$ η' fields. (The choice $c = 1/\sqrt{n-k}$ reproduces the canonical normalization.) Similarly, the contraction of a super-η' with some combination of unquenched quark fields is the same as the contraction of an $SU(n-k)$ η' with the same combination, since only the unquenched quarks in the super-η' will contribute.

Theorem III relies on the fact that the infrared divergences that have been found [3-5] in the quenched theory arise from the two-hairpin diagram (Fig. 1b), which gives a double pole in the η' propagator. In section 3, we calculate the propagator in the neutral meson sector in partially quenched chiral perturbation theory, and show that the offending double poles can only arise when a mass scale is fully quenched. Here, we make this result plausible by examining the degenerate $SU(n|k)$ theory. It is not difficult to argue that, when there is only one quark mass scale, double poles arise only when the theory is completely quenched, *i.e.*, when $k = n$.

Consider the propagator of the η' of the degenerate $SU(n|k)$ theory (constructed from the n quarks only, with no pseudoquarks). In a normal unquenched theory, the two-hairpin contribution to the propagator would include diagrams with arbitrary numbers of virtual quark bubbles between the two hairpins. The set of all these diagrams, together with the straight-through diagram, is a geometric series which can be summed to a simple pole, with an η' mass shifted away from the common meson multiplet mass by the usual singlet contribution. In a partially quenched theory, the sum over bubbles is still present, and differs only by an overall normalization (coming from the counting of the *valence* loops in the hairpin) relative to the case where only the unquenched quarks are allowed. Therefore, the two-hairpin diagram has an "incorrect" normalization relative to the straight-through diagram, so the full propagator will not just be a simple pole with a singlet mass term added. We can correct for this mismatch by adding and subtracting the proper amount of straight-through diagram. The complete η' propagator will then be a sum of two simple poles, one with mass equal to the common meson multiplet mass (from the subtracted piece of the straight-through diagram) and one with a shifted mass which includes the singlet contribution. Since there are no double poles, there will be no unusual infrared divergences. The only exception occurs for $k = n$, when there are no bubbles to sum. Then the complete η' propagator is just the sum of two terms: a double pole from two-hairpin diagram and a single pole from the straight-through diagram [3,4].

An example which illustrates several of the ideas discussed above is the construction, in a degenerate $SU(n|k)$ theory, of an $SU(n-k)$ η' out of the diagrams for the $SU(n)$ η'. Let both of these particles be described by canonically normalized fields:

$$\begin{aligned} \eta'_{SU(n-k)} &= \frac{1}{\sqrt{n-k}} \sum_{i=1}^{n-k} \overline{q}_i \gamma_5 q_i , \\ \eta'_{SU(n)} &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \overline{q}_i \gamma_5 q_i \ . \end{aligned} \tag{2.2}$$

The straight-through diagram of Fig. 1a is then clearly identical for the $\eta'_{SU(n)}$ and the $\eta'_{SU(n-k)}$, since the factor of n or $n-k$ from flavor counting in the loop cancels against the field normalization factors. The two-hairpin diagram (Fig. 1b) is however normalized differently for the $\eta'_{SU(n)}$ and the $\eta'_{SU(n-k)}$, since there are now two flavor loops. To get the $\eta'_{SU(n-k)}$ two-hairpin from the $\eta'_{SU(n)}$ two-hairpin, one must multiply the latter by $(n-k)/n$. This holds irrespective of the number of virtual loops in the two-hairpin diagram. Note that each virtual bubble is normalized the same in both cases, since we are always working in a $SU(n|k)$ theory with a net total of $n-k$ flavors in virtual loops. The difference in normalization arises only from the valence (hairpin) loops.

We may thus obtain the correct $\eta'_{SU(n-k)}$ propagator in an $SU(n|k)$ theory from the $\eta'_{SU(n)}$ propagator in that same theory by making a simple readjustment of the relative weights of the diagrams. This will prove useful when we discuss staggered fermions.

3. Proof of Theorem III

We begin by writing down the lagrangian for the $SU(n|k)$ theory. Define the $(n+k)\times(n+k)$ hermitian field Φ by

$$\Phi \equiv \begin{pmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{pmatrix}, \tag{3.1}$$

where ϕ is the $n\times n$ matrix of ordinary mesons made from the n ordinary quarks and their antiquarks, $\tilde{\phi}$ is the corresponding $k\times k$ matrix for pseudoquark mesons, and χ is a $k\times n$ matrix of mesons made from a pseudoquark and an ordinary antiquark. The unitary field Σ is then defined as

$$\Sigma \equiv \exp(2i\Phi/f)\ , \tag{3.2}$$

with f the tree-level pion decay constant. The $(n+k)\times(n+k)$ quark mass matrix is given by

$$\mathcal{M}_{ij} = m_i\delta_{ij}\ , \tag{3.3}$$

where, as discussed in the previous section, the masses m_i for $i=1,\ldots,n$ are arbitrary, and we take the pseudoquark masses equal to the first k quark masses: $m_{n+j}\equiv m_j$ for $j=1,\ldots,k$.

The euclidean lagrangian is then

$$\begin{aligned}\mathcal{L} =& V_1(\Phi_0) str(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger) - V_2(\Phi_0) str(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) \\ &+ V_0(\Phi_0) + V_5(\Phi_0)(\partial_\mu\Phi_0)^2,\end{aligned} \tag{3.4}$$

where the functions V_i can be chosen to be real and even by making use of the freedom allowed by field redefinitions [3]. We choose $\Phi_0 = str(\Phi)$ which corresponds to $c = 1$ in eq. (2.1). Since the two-hairpin-like interactions between neutral mesons must have no dependence on n or k at tree level, this choice of c guarantees that the parameters in the expansion of the V_i are n- and k-independent at tree level.

For the purposes of this section we just need the quadratic terms in (3.4). We define

$$\begin{aligned} V_1(0) &\equiv \frac{f^2}{8} , \\ V_2(0) &\equiv v \left(= \frac{f^2 m_{\pi^+}^2}{4(m_u + m_d)} \right) , \\ V_0''(0) &\equiv \frac{\mu^2}{3} , \\ V_5(0) &\equiv \frac{\alpha}{6} . \end{aligned} \tag{3.5}$$

Note that $V_5(0)$ and $V_0''(0)$ are not the same as in ref. [3] because of the different choice of normalization for Φ_0.

We are now in the position to prove theorem III. In the case of arbitrary quark mass the simplest approach is just to calculate the neutral-meson propagator explicitly in tree approximation. We work in the basis of the states U_i, $i = 1, \ldots, n + k$ corresponding to $\overline{u}u, \overline{d}d, \overline{s}s, \ldots$, and their pseudoquark counterparts. From eqs.(3.4) and (3.5), the neutral inverse propagator in momentum space is

$$G_{ij}^{-1} = \delta_{ij}(p^2 + M_i^2)\epsilon_i + \frac{\mu^2}{3}\epsilon_i\epsilon_j, \tag{3.6}$$

where $M_i^2 \equiv 8vm_i/f^2$, we have taken $\alpha = 0$ (it is easy to reinstate later on by the substitution $\mu^2 \to \mu^2 + \alpha p^2$), and ϵ_i is defined by

$$\epsilon_i = \begin{cases} +1, & \text{for } 1 \leq i \leq n \\ -1, & \text{for } n+1 \leq i \leq n+k \end{cases} . \tag{3.7}$$

It is straightforward to invert (3.6), either by expanding in powers of μ^2 or by guessing the form of the inverse and fixing the coefficients by $GG^{-1} = 1$. We have

$$G_{ij} = \frac{\delta_{ij}\epsilon_i}{p^2 + M_i^2} - \frac{\mu^2/3}{(p^2 + M_i^2)(p^2 + M_j^2)F(p^2)} , \tag{3.8}$$

where

$$F(p^2) \equiv 1 + \frac{\mu^2}{3}\sum_{i=1}^{n+k}\frac{\epsilon_a}{p^2 + M_i^2} = 1 + \frac{\mu^2}{3}\sum_{r=k+1}^{n}\frac{1}{p^2 + M_r^2} . \tag{3.9}$$

The last equality in (3.9) follows from the fact that the pseudoquark masses have been chosen equal to the first k quark masses.

Theorem III now follows by examination of eq. (3.8). We show in the Appendix that $F(p^2)$ has no double zeros, so no double poles in G can arise from F. Therefore, the only way there can be a double pole in G is for $M_i = M_j$, for some i, j (this is of course trivially satisfied for $i = j$), and $M_j \neq M_r$, for all j between $k+1$ and n. Since M_r, $k+1 \leq r \leq n$, are just the masses of the neutral mesons composed of unquenched quarks, the latter condition implies that double poles occur at mass M_j if and only if quarks of the corresponding mass are completely quenched. This is just the content of theorem III.

It is instructive to examine eq. (3.8) in the degenerate limit ($M_i \equiv M$ for all i). For $k \neq n$, we have

$$G_{ij} = \frac{\delta_{ij}\epsilon_i - 1/(n-k)}{p^2 + M^2} + \frac{1/(n-k)}{p^2 + M^2 + (n-k)\mu^2/3} \, . \tag{3.10}$$

This clearly illustrates a result of section 2: that the degenerate propagator (for $k \neq n$) is the sum of two simple poles, one with mass equal to the common meson multiplet mass and one with a shifted mass which includes the singlet contribution. Note that for $k = n$ one sees immediately from eqs. (3.8) and (3.9) that there are always double poles in G for $i = j$.

4. Examples

We would like to demonstrate the theorems in some explicit examples. First, we will consider the case of completely degenerate quark masses, with n normal quarks and k pseudoquarks. We have calculated the self-energies of the pion and the super-η' to one loop. The euclidean-space pion self-energy is

$$\begin{aligned}\Sigma_\pi(p) = &-\frac{2}{3}(n-k)\left(p^2 + m_\pi^2\right) I(m_\pi^2) + \frac{2}{(n-k)} m_\pi^2 \left(I(m_\pi^2) - \frac{I(m_{\eta'}^2)}{1 + \frac{1}{3}\alpha(n-k)} \right) \\ &+ 4(n-k)\left(V_1''(0)p^2 + \frac{f^2 V_2''(0)}{8v} m_\pi^2 \right) \frac{I(m_{\eta'}^2)}{1 + \frac{1}{3}\alpha(n-k)},\end{aligned} \tag{4.1}$$

where $m_\pi^2 = 8vm/f^2$, and

$$I(m^2) = \frac{1}{f^2} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2}. \tag{4.2}$$

For the purpose of this paper we do not need to specify how we regulate such integrals. $m_{\eta'}$ is the mass of the η' in the $SU(n-k)$ theory, and is given by

$$m_{\eta'}^2 = \frac{\frac{1}{3}(n-k)\mu^2 + m_\pi^2}{1 + \frac{1}{3}\alpha(n-k)}. \tag{4.3}$$

Theorem I is clearly obeyed by this result: the self-energy is a function of $n-k$ only, and therefore is equal to the pion self-energy computed in a theory with $n-k$ normal quarks and no pseudoquarks.

For $k < n$ the above results only have chiral logarithms of the standard type, arising from integrals over single pole propagators. On the other hand, for $k = n$ the result is

$$\Sigma_{\pi,k=n}(p) = \frac{2}{3}\frac{m_\pi^2}{f^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{\mu^2 + \alpha q^2}{(q^2 + m_\pi^2)^2}. \tag{4.4}$$

This contributes a term to the pion mass which goes like $\mu^2 m_\pi^2 \log m_\pi^2$, unlike the usual $m_\pi^4 \log m_\pi^2$. It is an example of the "pathological terms" previously seen in quenched calculations.[2] This is a special case of theorem III: there is a pathology as $m_\pi \to 0$ which comes from a double pole and arises only for $k = n$, in which case the mass scale m is fully quenched.

To demonstrate theorem II, we have to compute the self-energy for the super-η', Φ_0. On the external lines, we choose $c = 1/\sqrt{n-k}$ in eq. (2.1), so that for $k = 0$ we are just calculating the conventionally normalized η' self energy in an ordinary unquenched theory. (Recall, however, that Φ_0 in the potentials $V_i(\Phi_0)$ is normalized with $c = 1$.) For the degenerate case, the result is

$$\begin{aligned}\Sigma_{\Phi_0}(p) =& \frac{-2m_\pi^2}{n-k}\left(\left[(n-k)^2 - 1\right] I(m_\pi^2) + \frac{I(m_{\eta'}^2)}{1+\frac{1}{3}\alpha(n-k)}\right)\\ &+ \frac{f^2 V_2''(0)}{2v}(n-k)m_\pi^2\left(\left[(n-k)^2 - 1\right] I(m_\pi^2) + \frac{6I(m_{\eta'}^2)}{1+\frac{1}{3}\alpha(n-k)}\right)\\ &- 4V_1''(0)(n-k)\left(\left[(n-k)^2 - 1\right] m_\pi^2 I(m_\pi^2) + (m_{\eta'}^2 - p^2)\frac{I(m_{\eta'}^2)}{1+\frac{1}{3}\alpha(n-k)}\right)\\ &+ f^2\left(\frac{1}{2}V_0''''(0) + V_5''(0)(p^2 - m_{\eta'}^2)\right)(n-k)^2\frac{I(m_{\eta'}^2)}{1+\frac{1}{3}\alpha(n-k)}.\end{aligned} \tag{4.5}$$

---

[2] In other quantities such as $\langle\overline{\psi}\psi\rangle$ or f_K/f_π such pathologies can lead to actual infrared divergences as a quark mass is taken to zero, see refs. [3-5].

Again, for $k < n$ we see that this is only a function of $n-k$, and therefore equal to the self-energy of the η' in the $SU(n-k)$ theory. For $k=n$ the normalization $c = 1/\sqrt{n-k}$ is clearly inappropriate, and we should multiply (4.5) through by $(n-k)$. The expression then vanishes when $n=k$ (note that in that case $m_\pi = m_{\eta'}$), consistent with our expectation that the Φ_0 propagator (with finite c) vanishes in the fully quenched theory [3].

As a final example, we consider the "doublet $SU(2n|2k)$" theory, in which we have n quarks and k pseudoquarks with mass m_u, and n quarks and k pseudoquarks with mass m_d. We will only present that part of the one loop pion self-energy which comes from the 4-meson vertex proportional to $V_1(0)$, since the full expression is quite cumbersome. All our conclusions hold separately for the contribution from each vertex to the self-energy, since the parameters multiplying these vertices are free. The result is (here we set $\alpha = 0$ for simplicity)

$$\begin{aligned}
\Sigma_\pi^{V_1(0)}(p) = &-\frac{1}{3f^2}\int \frac{d^4q}{(2\pi)^4}(q^2+p^2)\Bigg[(n-k)\left(\frac{1}{q^2+m_U^2}+\frac{1}{q^2+m_D^2}+\frac{2}{q^2+\overline{m}^2}\right) \\
&-\frac{1}{n-k}\left(\frac{1}{q^2+m_U^2}+\frac{1}{q^2+m_D^2}\right) \\
&+\frac{2}{n-k}\left(\cos^2\theta\frac{1}{q^2+m_+^2}+\sin^2\theta\frac{1}{q^2+m_-^2}\right)\Bigg].
\end{aligned} \tag{4.6}$$

(The quartic divergence present in eq. (4.6) is cancelled in the total self-energy by a term coming from the measure of the path integral.) In this expression,

$$\begin{aligned}
m_U^2 &= \frac{8m_u v}{f^2}, \\
m_D^2 &= \frac{8m_d v}{f^2}, \\
\overline{m}^2 &= \frac{4(m_u+m_d)v}{f^2} = \frac{1}{2}(m_U^2+m_D^2), \\
m_\pm^2 &= (\overline{m}^2+\frac{1}{3}(n-k)\mu^2) \mp \sqrt{\frac{1}{9}(n-k)^2\mu^4+\frac{1}{4}(m_U^2-m_D^2)^2},
\end{aligned} \tag{4.7}$$

and

$$\sin^2\theta = \frac{n-k}{12}\frac{\mu^2(m_U^2-m_D^2)^2}{(m_-^2-m_U^2)(m_-^2-m_D^2)\sqrt{\frac{1}{9}(n-k)^2\mu^4+\frac{1}{4}(m_U^2-m_D^2)^2}}. \tag{4.8}$$

The five different masses which appear in eq. (4.6) correspond to the various meson masses which appear in the $SU(2(n-k))$ theory. In the flavor off-diagonal sector, m_U and m_D correspond to mesons of types $\overline{u}_i u_j$ and $\overline{d}_i d_j$ with $i \neq j$, and $\overline{m}$ to $\overline{d}u$ or $\overline{u}d$. In the flavor diagonal sector, there are $2(n-k-1)$ π^0-like mesons with masses m_U and m_D, and two other neutral mesons with masses $m_\pm$ due to the singlet-nonsinglet mixing which occurs for $m_u \neq m_d$. θ is the mixing angle between these two latter neutral mesons. In the case that $n-k=1$, the coefficients of the m_U^2 and m_D^2 poles in eq. (4.6) vanish, consistent with the meson spectrum of the $SU(2)$ theory.

For $k=n$, we have

$$\Sigma^{V_1(0)}_{\pi,k=n}(p) = \frac{\mu^2}{9f^2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}(q^2+p^2)\left(\frac{1}{q^2+m_U^2}-\frac{1}{q^2+m_D^2}\right)^2 , \tag{4.9}$$

which is independent of n since results in a fully quenched theory must depend only on the valence quarks. Eq. (4.9) agrees for $u \to s$ with the result we obtained in computing the one loop corrections to the kaon mass (*cf.* ref. [3]).

5. Application to Staggered Fermions

In this section we will apply some of the results obtained in the previous sections to lattice QCD with staggered fermions. In the scaling region, this theory describes QCD with four quark flavors, which can be given nondegenerate masses by using nonlocal mass terms [6,7]. If one would like to consider QCD with two flavors, one can use the so-called reduced staggered fermion formalism [8,9], which however leads to a complex fermion determinant [9]. Also the reduced staggered fermion action does not possess any continuous chiral invariance, unlike "normal" staggered fermions. An alternative is to consider normal staggered fermions and define a two-flavor theory by taking the square root of the determinant [10]. This corresponds to quenching two of the four flavors. We therefore expect that the low energy meson effective theory will be described by $SU(4|2)$ chiral perturbation theory. For this to work, the masses need to be at least pairwise degenerate.

If only a single site mass term is used, the staggered fermion determinant with degenerate quark masses is positive [11]. Since the continuum limit is a degenerate four-flavor theory with a determinant which is the fourth power of a one-flavor determinant, one expects that taking the positive square root of the staggered fermion determinant leads to the desired determinant for the continuum two-flavor theory.

If nonlocal mass terms are used, the determinant is not positive in general. However, the continuum determinant for each flavor is (formally) positive, so one might expect that with staggered fermions close enough to the continuum limit, no problem arises in taking the square root.

In this section, we will consider the definition of two-flavor meson operators in the mass degenerate two-flavor theory obtained from the degenerate four-flavor theory in which the square root of the determinant is taken. Theorem I tells us that we can obtain the two-flavor unquenched theory in this way, and that no problems are to be expected from taking the square root. For nonsinglet mesons no tuning of the operators is required because one may use the same operators as in the four-flavor theory.

In general, however, one will need to tune the staggered hadron operators in order to project out the various continuum hadronic states of interest [6]. For example, tuning will be necessary in order to have more than one different quark mass within the staggered fermion formalism [6,12]. In particular, one expects that the definition of an operator for the $\eta'_{SU(2)}$ in the four-flavor theory will require tuning, even with degenerate quark masses. What we wish to show here is that nevertheless two ways exist for choosing a mass matrix and a meson operator which do not require tuning of the operator in order to define a pure $\eta'_{SU(2)}$ in the four-flavor theory. The first method consists of applying theorem II, whereas the second method makes use of a peculiarity of nonlocal staggered fermion mass terms.

We will start by reviewing some facts about renormalization for staggered fermions. A general mass term for staggered fermions is given by [6]

$$
\begin{aligned}
S_{\mathrm{mass}} = &\sum_{x} m\overline{\chi}(x)\chi(x) + \sum_{xy} m_\mu \overline{\chi}(x)E_\mu(x,y)\chi(y) - \tfrac{1}{2}i\sum_{xyz} m_{\mu\nu}\overline{\chi}(x)E_\mu(x,y)E_\nu(y,z)\chi(z) \\
&- \tfrac{1}{6}i\sum_{wxyz} m^5_\mu \epsilon_{\mu\alpha\beta\gamma}\overline{\chi}(w)E_\alpha(w,x)E_\beta(x,y)E_\gamma(y,z)\chi(z) \\
&- \tfrac{1}{24}\sum_{vwxyz} m^5 \epsilon_{\alpha\beta\gamma\delta}\overline{\chi}(v)E_\alpha(v,w)E_\beta(w,x)E_\gamma(x,y)E_\delta(y,z)\chi(z).
\end{aligned}
\tag{5.1}
$$

The operator E is defined through

$$
\sum_{xy} m_\mu \overline{\chi}(x)E_\mu(x,y)\chi(y) = \tfrac{1}{2}\sum_{z\mu} m_\mu \zeta_\mu(z)[\overline{\chi}(z)U_\mu(z)\chi(z+\hat{\mu}) + \overline{\chi}(z+\hat{\mu})U^\dagger_\mu(z)\chi(z)], \tag{5.2}
$$

where the ζ_μ are certain site-dependent sign factors (*cf.* ref. [6]). $m_{\mu\nu}$ is taken to be antisymmetric.

This mass term leads to the following mass matrix M for the four flavors that emerge in the continuum limit:

$$M = m + m_\mu \xi_\mu + \tfrac{1}{2} m_{\mu\nu}(-i\xi_\mu\xi_\nu) + m^5_\mu i\xi_\mu\xi_5 + m^5\xi_5. \tag{5.3}$$

The 4×4 ξ-matrices form a representation of the Clifford algebra $\xi_\mu\xi_\nu + \xi_\nu\xi_\mu = 2\delta_{\mu\nu}$, and are identified with $SU(4)$ flavor generators in the continuum limit. We will denote the terms in eq. (5.3) with scalar (S), vector (V), tensor (T), axial vector (A) and pseudoscalar (P) respectively. This expression for M can be derived from the fact that each shift in the μ-direction of the field χ, accompanied by a multiplication with ζ_μ, corresponds to a multiplication of the continuum four-flavor Dirac field ψ by the matrix ξ_μ:

$$\zeta_\mu(x)\chi(x+\hat{\mu}) \to \xi_\mu\psi(x). \tag{5.4}$$

It can be shown that this form of the mass matrix is stable under renormalization, in the sense that the coefficients m, m_μ, ... will only receive multiplicative renormalizations, one for each tensor structure in eq. (5.3). This was explicitly demonstrated to one loop, and supplemented with more general symmetry arguments, in ref. [6]. Due to the presence, in the massless theory, of shift symmetries and a continuous chiral symmetry (the so-called $U(1)_\epsilon$ symmetry) there are no additive counterterms. Note that the mass matrix M needs to be diagonalized in order to determine what the mass eigenstates are.

Let us first consider the simplest possible mass matrix, by choosing only the single site mass m in eq. (5.1) to be nonzero, corresponding to four degenerate flavors. In this case, the simplest operator for an $\eta'_{SU(4)}$ will be

$$\begin{aligned}\eta'(x) \propto \overline{\chi}(x)\epsilon(x)\zeta_1(x)\zeta_2(x+\hat{1})\zeta_3(x+\hat{1}+\hat{2})\zeta_4(x+\hat{1}+\hat{2}+\hat{3})\chi(x+\hat{1}+\hat{2}+\hat{3}+\hat{4})\\ + \text{sum over all permutations on the directions } \hat{1},\ \hat{2},\ \hat{3} \text{ and } \hat{4},\end{aligned} \tag{5.5}$$

which in the continuum limit corresponds to the operator $\overline{\psi}\gamma_5\psi$ [6], where ψ is a continuum Dirac field with four flavor components. In eq. (5.5), the lattice gauge fields are implicit.

In this basis, an $\eta'_{SU(2)}$ would be created by the continuum operator

$$\eta'^{\rm cont}_{SU(2)} = \overline{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma_5 \psi. \tag{5.6}$$

Clearly, in order to construct a staggered operator with this continuum limit, we need the operator S to get a nonzero trace because the $\eta'_{SU(2)}$ flavor matrix in eq. (5.6) has a nonvanishing trace, and V, T, A and P are all traceless. In addition, we need an operator of the type V, T, A and P, since the matrix contains two zero eigenvalues. The fact that these operators renormalize differently from S leads to the need to tune their relative coefficient. We conclude that with a single site mass term no explicit $\eta'_{SU(2)}$ operator can be constructed in the four-flavor staggered theory without tuning. The only way to avoid tuning in this case, is to compute the diagrams for the $\eta'_{SU(4)}$ (eq. (5.5)), and adjust the relative coefficients of the straight-through and the two-hairpin diagrams as discussed at the end of section 2.

Actually, the special properties of the tensor operator make it possible to construct an $\eta'_{SU(2)}$ without tuning in a different way. To discuss this, we will choose an explicit representation of the ξ-matrices:

$$\xi_i = \sigma_i \otimes \tau_1, \quad \xi_4 = \tau_2, \quad \xi_5 = \tau_3. \tag{5.7}$$

In this case, it is necessary to choose a mass term of the tensor type. For definiteness we choose

$$M_0 = m(-i\xi_1\xi_2) = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & -m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & -m \end{pmatrix}, \tag{5.8}$$

which corresponds again to four flavors with a degenerate mass m. The minus signs can be removed by a nonanomalous chiral transformation. The $\eta'_{SU(4)}$ with this mass matrix is

$$\eta'^{\rm cont}_{SU(4)} \propto \overline{\psi}(-i\xi_1\xi_2)\gamma_5\psi = \overline{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \gamma_5 \psi\ . \tag{5.9}$$

Projecting to $SU(2)$, we get for the $\eta'_{SU(2)}$

$$\eta'^{\rm cont}_{SU(2)} \propto \overline{\psi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma_5 \psi = \overline{\psi}(-i\xi_1\xi_2 - i\xi_3\xi_4)\gamma_5\psi. \tag{5.10}$$

Unlike the previous case, this $\eta'_{SU(2)}$ flavor matrix is now traceless, which allows us to write it as a sum of two tensor terms. The minus sign which appears in this equation is removed by the same chiral tranformation that removes the sign in the mass matrix, eq. (5.8). The relevant observation here is that this $\eta'_{SU(2)}$ flavor matrix is not traceless with respect to the mass matrix, i.e. $\mathrm{tr}\,(M_0(-i\xi_1\xi_2 - i\xi_3\xi_4)) \neq 0$. This follows from the fact that the mass matrix defines what the flavor symmetries are (in the continuum limit). If ψ_L and ψ_R transform under $SU(4)_L \otimes SU(4)_R$ as

$$\psi_L \to V_L\psi_L, \quad \psi_R \to V_R\psi_R, \tag{5.11}$$

the condition for invariance is

$$V_L^\dagger M V_R = M. \tag{5.12}$$

With a degenerate mass matrix as in eq. (5.8), the symmetry group is $SU(4)$ (in the continuum limit). If a chiral transformation is performed to remove the minus signs in eq. (5.8), eq. (5.12) and the trace condition take on their usual form.

As mentioned above, the $\eta'_{SU(2)}$ of eq. (5.9) is now constructed from two tensor operators rather than one scalar and one of some other type. Since all tensor operators get renormalized in the same way, no tuning is needed here. The price, however, is the use of a tensor mass term, which would make this approach awkward for standard simulations. Also, in the case of a tensor mass, in general the staggered fermion determinant is not positive (*cf.* the introduction to this section). Using the $\eta'_{SU(4)}$ and readjusting the relative weight of the diagrams by hand, as explained in section 2, will be preferable in most cases.

6. Conclusion

In this paper our investigations of ChPT in the quenched approximation of QCD are extended to theories in which only some of the quarks are quenched.

The results are formulated in three theorems. Two of them state that in the subsector with unquenched valence quarks the theory is equivalent to an unquenched theory with the number of flavors equal to the number of unquenched quarks in the partially quenched theory. The super-η' of the partially quenched theory is equivalent to the η' of this unquenched theory.

The third theorem deals with the existence of infrared divergences due to the double pole in the quenched η' twopoint function [3-5]. Such divergences only arise if a particular quark mass scale is completely quenched. They do not show up in correlation functions with only partially quenched or unquenched quarks on the external lines.

Some one-loop calculations serve as explicit examples of these results. Moreover, we apply the $n = 4$, $k = 2$ case to staggered fermion QCD with a single site mass term, in which the square root of the fermion determinant is taken in order to yield two-flavor QCD. Our analysis shows that this technique is valid within ChPT, and that the super-η' of the $SU(4|2)$ theory (or equivalently the $SU(4)$ η' with by-hand reweighting of diagrams) can be used to measure $SU(2)$ η' correlation functions without any of the fine tuning which is often necessary for staggered fermions.

Finally, we have shown that another two-flavor η' operator, not based on the super-η', exists for which no fine tuning is needed if one employs staggered fermions with a so-called tensor mass.

Acknowledgements

We are greatful to Doug Toussaint for inspiring this project by suggesting that our chiral techniques could be used to examine the determinant square-root method for staggered fermions. We also thank Mike Ogilvie for very useful discussions, and Carl Bender for the proof of the lemma presented in the appendix. Part of this work was carried out at Los Alamos National Laboratory and UC Santa Barbara. M.G. would like to thank Rajan Gupta and the Theory Division of LANL, and both of us would like to thank the UCSB Physics Department, and in particular Bob Sugar, for hospitality.

C.B. and M.G. are supported in part by the Department of Energy under grant #DOE-2FG02-91ER40628.

Appendix

In this appendix we present a proof[3] of the lemma that we used in section 3. The lemma states that the function

$$f(z) = 1 + \sum_{i=1}^{k} \frac{1}{z - \alpha_i} \tag{6.1}$$

has no double zeros if all the α_i are real. The proof will be by contradiction. So let us assume that $f(z)$ has a double zero at $z = \beta$. First, $f(z)$ diverges when z is equal to any of the α_i, so we can assume that $\beta \neq \alpha_i$ for all i. Now define $z' = z - \beta$. Then $f(z')$, which is given by

$$f(z') = 1 + \sum_{i=1}^{k} \frac{1}{z' - \alpha_i'} \tag{6.2}$$

with $\alpha_i' = \alpha_i - \beta$, now has a double zero at $z' = 0$, with all $\alpha_i \neq 0$. Note that the α_i' are not necessarily real, but can have a common imaginary part. From now on we will drop the primes on z and α_i and assume that $f(z)$ has a double zero at $z = 0$. f can be written as

$$f(z) = \frac{P_k(z)}{\prod_{i=1}^{k}(z - \alpha_i)}, \tag{6.3}$$

where $P_k(z)$ is a polynomial in z of degree k:

$$P(z) = z^k + \ldots + \prod_{i=1}^{k}(-\alpha_i)\left[-\sum_{i=1}^{k}\frac{1}{\alpha_i} + \sum_{i\neq j}\frac{1}{\alpha_i\alpha_j}\right] z + \prod_{i=1}^{k}(-\alpha_i)\left[1 - \sum_{i=1}^{k}\frac{1}{\alpha_i}\right]. \tag{6.4}$$

If $f(z)$ has a double zero at $z = 0$, the constant term and the coefficient of the linear term in $P(z)$ have to vanish, *i.e.*,

$$\sum_{i=1}^{k}\frac{1}{\alpha_i} = 1, \qquad \sum_{i=1}^{k}\frac{1}{\alpha_i} = \sum_{i\neq j}\frac{1}{\alpha_i\alpha_j} = \left(\sum_{i=1}^{k}\frac{1}{\alpha_i}\right)^2 - \sum_{i=1}^{k}\frac{1}{\alpha_i^2}. \tag{6.5}$$

From the first of these equations one concludes that the common imaginary part of the α_i has to vanish. Therefore the α_i have to be real (in other words, the original double zero

[3] This proof is due to C. Bender.

would have to be on the real axis). Substituting the first equation into the second, we then conclude that

$$\sum_{i=1}^{k} \frac{1}{\alpha_i^2} = 0, \tag{6.6}$$

which has no solution for real α_i. This completes the proof that the function $f(z)$ has no double zeros anywhere in the complex plane.

References

[1] A. Morel, J. Physique **48**,111(1987).

[2] S.R. Sharpe, Phys. Rev. **D41**, 3233 (1990); Nucl. Phys. **B**(Proc.Suppl.) **17**, 146 (1990); G. Kilcup *et al.*, Phys. Rev. Lett. **64**, 25 (1990); S.R. Sharpe, DOE/ER/40614-5, to be published in *Standard Model, Hadron Phenomenology and Weak Decays on the Lattice*, ed. G. Martinelli, World Scientific.

[3] C.W. Bernard and M.F.L. Golterman, Phys. Rev. **D46**, 853 (1992); Nucl. Phys. **B**(Proc.Suppl.) **26**, 360 (1992).

[4] C.W. Bernard and M.F.L. Golterman, Nucl. Phys. **B**(Proc.Suppl.) **30**, 217 (1993).

[5] S.R. Sharpe, Phys. Rev. **D46**, 3146 (1992); Nucl. Phys. **B**(Proc.Suppl.) **30**, 213 (1993).

[6] M.F.L. Golterman and J. Smit, Nucl. Phys. **B245**, 61 (1984).

[7] M. Göckeler, Phys. Lett. **B142**, 197 (1984).

[8] H.S. Sharatchandra, H.J. Thun and P. Weisz, Nucl. Phys. **B192**, 205 (1981).

[9] C. van den Doel and J. Smit, Nucl. Phys. **B228**, 122 (1983).

[10] E. Marinari, G. Parisi and C. Rebbi, Nucl. Phys. **B190**, 734 (1981).

[11] J. Smit and J. Vink, Nucl. Phys. **B286**, 485 (1987).

[12] M.F.L. Golterman and J. Smit, Phys. Lett. **B140**, 392 (1984).