B Mixing on the Lattice: f_B, f_{B_s} and Related Quantities

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Abstract. The MILC collaboration computation of heavy-light decay constants is described. Results for f_B, f_{B_s}, f_D, f_{D_s} and their ratios are presented. These results are still preliminary, but the analysis is close to being completed. Sources of systematic error, both within the quenched approximation and from quenching itself, are estimated, although the latter estimate is rather crude. A sample of our results is: $f_B = 153 \pm 10\ ^{+36}_{-13}\ ^{+13}_{-0}$ MeV, $f_{B_s}/f_B = 1.10 \pm 0.02\ ^{+0.05}_{-0.03}\ ^{+0.03}_{-0.02}$, and $f_B/f_{D_s} = 0.76 \pm 0.03\ ^{+0.07}_{-0.04}\ ^{+0.02}_{-0.01}$, where the errors are statistical, systematic (within the quenched approximation), and systematic (of quenching), respectively. The largest source of error comes from the extrapolation to the continuum. The second largest source is the chiral extrapolation. At present, the central values are based on linear chiral extrapolations; a shift to quadratic extrapolations would for example raise f_B by ≈ 20 MeV and thereby make the error within the quenched approximation more symmetric.

INTRODUCTION

B_d-$\bar{B}_d$ mixing offers a way to determine the CKM matrix element V_{td}. Indeed, the mixing parameter x_d, defined as $\Delta M_{B_d}/\Gamma_{B_d}$, is given in the Standard Model by [1]

[1] presented by C. Bernard at *b20: Twenty Beautiful Years of Bottom Physics*, Illinois Institute of Technology, June 29-July 2, 1997

$$x_d = \tau_{B_d} \frac{G_F^2 M_W^2}{6\pi^2} \eta_B S(x_t) M_{B_d} \xi_{B_d}^2 |V_{td}|^2 \ , \qquad (1)$$

where $\eta_B S(x_t)$ is perturbatively known, $V_{tb} \cong 1$ is assumed, and

$$\xi_{B_d} \equiv f_B \sqrt{B_B} \ , \qquad (2)$$

with f_B the decay constant of the B_d meson and B_B the corresponding "bag parameter." Since x_d has been measured experimentally, an evaluation of the nonperturbative quantity ξ_{B_d} will determine V_{td}. Similarly, a measurement or bound on the corresponding mixing parameter x_s for B_s mesons will give information about V_{ts} if ξ_{B_s} is known, or about $|V_{td}/V_{ts}|$, given ξ_{B_s}/ξ_{B_d}.

The lattice offers a way to compute quantities like f_B and B_B from first principles. Here, we present a nearly completed computation by the MILC collaboration of the decay constants f_B, f_{B_s}, f_D, f_{D_s}, and their ratios. A calculation of the quantities B_B and B_{B_s}/B_B, in the static limit for the heavy quark, is also in progress; some preliminary results are described in Ref. [2].

SOURCES OF ERROR

A key issue in any lattice computation is the reliability of systematic error estimates. In order to make these estimates, we use a range of lattices, both quenched (*i.e.*, negecting virtual quark loop effects) and unquenched. The lattice spacing and volume are varied over as wide a range as practical. Lattices used are shown in Table 1. We discuss the sources of error in turn.

TABLE 1. Lattices used. The quark mass in virtual loops (m_{vir}) is given in lattice units; where it is absent the lattices are quenched. a is the lattice spacing, and ℓ is the spatial box size. Lattice Q is used for a finite size check only, and does not appear in the rest of the analysis. Lattice G was generated by HEMCGC; lattice F, by the Columbia group.

Name	β	m_{vir}	size	# confs.	a^{-1}(GeV)	ℓ(fm)
A	5.7	–	$8^3 \times 48$	200	1.3	1.2
B	5.7	–	$16^3 \times 48$	100	1.3	2.5
E	5.85	–	$12^3 \times 48$	100	1.7	1.4
C	6.0	–	$16^3 \times 48$	100	2.0	1.6
Q	6.0	–	$12^3 \times 48$	235	2.0	1.2
D	6.3	–	$24^3 \times 80$	100	3.1	1.6
H	6.52	–	$32^3 \times 100$	60	4.2	1.5
L	5.445	0.025	$16^3 \times 48$	100	1.3	2.5
N	5.5	0.1	$24^3 \times 64$	100	1.4	3.6
O	5.5	0.05	$24^3 \times 64$	100	1.5	3.2
M	5.5	0.025	$20^3 \times 64$	100	1.5	2.6
P	5.5	0.0125	$20^3 \times 64$	100	1.7	2.4
G	5.6	0.01	$16^3 \times 32$	200	2.1	1.5
F	5.7	0.01	$16^3 \times 32$	49	2.4	1.3

Statistics

The basic objects of interest are the quantum mechanical amplitudes for the propagation of mesons. Such amplitudes can be written as a functional integral, which in this case means the weighted sum over all possible paths for the quarks and over all possible configurations of the gluon field. Since it is impossible, for practical reasons, to include all such "paths," one resorts to a statistical sampling, and the answers therefore have statistical errors. Equivalently, one can say that we are doing a very large dimensional ($\sim 10^8$) integral by Monte Carlo importance-sampling methods. In practice, about 100 to 250 gluon field configurations are needed to reduce the statistical errors to well below the present systematic errors.

Isolating the State of Interest

Since we do not know, *a priori*, the B meson wave function, we cannot create a B directly on the lattice. Instead, we operate on the vacuum with an "interpolating field" which need only have the same quantum numbers as the B. This creates a superposition of all states with B quantum numbers: the B itself, radial excitations, $B+$ glueballs, *etc.*. However, on a Euclidean space lattice states develop in time as e^{-Et}, rather than e^{-iEt}. Therefore, after sufficient Euclidean time, all the higher energy states will have died away, leaving a pure B state.

The question then becomes: how long a time is "sufficient?" A standard approach is to define the "effective mass," $m_{\rm eff}$, as the instantaneous exponential rate of fall of the B meson propagator. As the higher energy states die away, the propagator will begin to fall like a pure exponential, and $m_{\rm eff}$ will approach the energy (or mass, for zero 3-momentum) of the B. Figure 1 shows a typical plot of $m_{\rm eff}$ *vs.* time for two propagators. (The propagator G_{sl} is needed because one wants to compute f_B, which is proportional to the amplitude for annihilating the B with an axial current.) For $t \gtrsim 12$, these effective masses show little systematic variation with time. However, because of statistical fluctuations, as well as possible lingering systematic effects, one gets slightly different results depending on which intervals (with $t \gtrsim 12$) are chosen to fit the propagators. The variation over the intervals (what we call the "fitting error") is added in quadrature with the naive statistical error before any further analysis.

Quark Mass Extrapolation/Interpolation

• Light-quark Extrapolation. We compute with light quark masses (m_q) in the range $m_s/3 \lesssim m_q \lesssim 2m_s$. This is because using physical m_u, m_d would (a) require too much computer time, (b) require too large a lattice, and (c) introduce spurious quenching effects [3].

The interpolation in m_q to the strange quark mass m_s introduces little systematic error (assuming one knows how to fix m_s on the lattice). However, the extrapolation in m_q to physical $m_{u,d}$ (the chiral extrapolation) is a significant source of error. That is because one doesn't know *a priori* the correct functional form of the extrapolation. For example, lowest order chiral perturbation theory predicts that m_π^2 is a linear function of quark mass. However, at the current statistical level, one sees small but significant deviations from linearity that are not well understood. In Fig. 2(a), the linear fit has very poor confidence level (taking into account the correlations in the data), despite the fact that it looks good to the eye.

The deviations from linearity could be due to unphysical effects such as the finite lattice spacing or the residual contamination by radial B excitations. Even the more "physical" cause (chiral logs or higher order analytic terms in chiral perturbation theory) are a source of spurious effects because quenched chiral logs are in general different from those in the full theory [3]. When we include virtual quark loops, the theory is at present only "partially quenched," since the sea quark mass is not tuned to be equal to the valence mass, and again spurious chiral logs will be present [4].

For these reasons, we presently fit quantities like m_π^2 to their lowest order chiral form, despite the poor confidence levels. The systematic error is estimated by repeating the analysis with quadratic (constrained) fits, as shown in Fig. 2(a). The systematic thus determined is $\leq 10\%$ for decay constants on all quenched data sets used to extrapolate to the continuum; usually it is

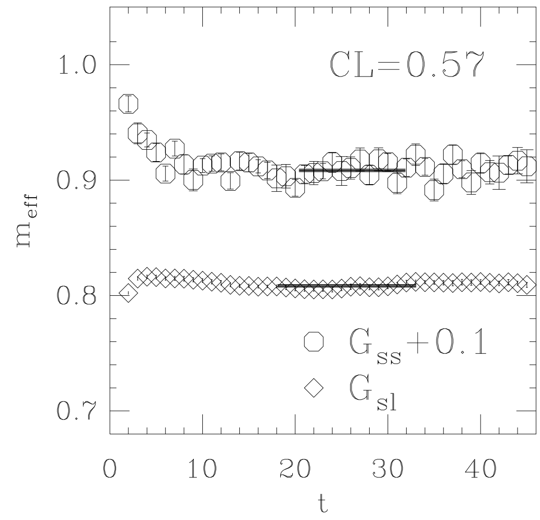

FIGURE 1. The effective mass $m_{\rm eff}$ *vs.* Euclidean time for two meson propagators (on lattice H) created by the same interpolating field. In G_{sl}, the meson is the annihilated by the axial current; in G_{ss}, by a second interpolating field. For G_{ss}, $m_{\rm eff}$ has been increased by 0.1 for clarity. The two propagators have been fit simultaneously in the ranges shown by the solid lines. The hopping parameters are 0.1474 and 0.125.

$\lesssim$5%. (After extrapolation to the continuum, the error is larger: 7% to 15% — see below.) For lattice F, this systematic is very large ($\gtrsim$50%), which may be due to the small volume, exacerbated by virtual quark loop effects. Lattice F is dropped from further analysis.

We emphasize that our reasons for choosing linear chiral fits for the central values are somewhat subjective, and it is possible that we will switch to quadratic fits in the final version of this work. To help us make the choice, we are studying a large sample of new lattices at $\beta = 5.7$, with large volumes up to 24^3. On this sample we have six light quark masses (as opposed to three for each of the lattices shown in Table 1) and have light-light mesons with nondegenerate as well as degenerate quarks.

• Heavy-quark Interpolation. Currently practical lattice spacings are in the range 1 GeV$\lesssim a^{-1} \lesssim$4 GeV. Thus $m_B a \gtrsim 1$, and it is difficult to simulate a B meson since its Compton wavelength is smaller than the lattice spacing. Our approach to this problem is to interpolate between heavy-quark masses that can be better simulated: infinite mass quarks treated by the static method [5] and normal, propagating quarks with masses lighter than the b.

For the propagating quarks, we correct for the gross lattice artifacts caused by $m_B a \sim 1$ with some of the techniques of Ref. [6]. We use the "EKM norm" and also shift from pole to kinetic heavy-quark mass at tadpole-improved tree level. There are further (but numerically less important) artifacts which we do not correct for; we expect them to be eliminated (or at least drastically reduced) by the continuum extrapolation ($a \to 0$) at the end.

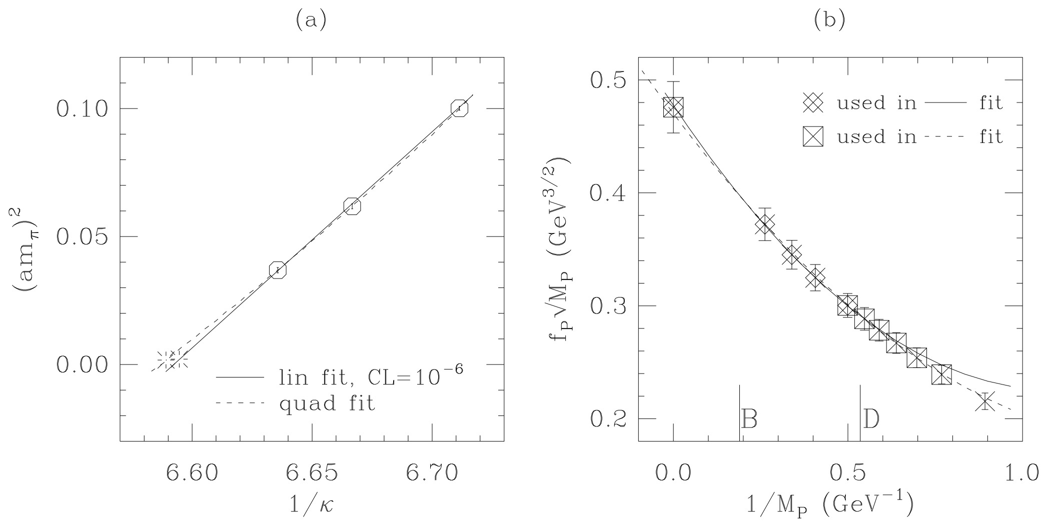

FIGURE 2. (a) The lattice m_π^2 *vs.* inverse hopping parameter ($1/\kappa \approx m_q +$ const.) on lattice D. Such fits are used to determine the value of κ which corresponds to the physical up or down quark mass. (b) $f_P M_P^{\frac{1}{2}}$ *vs.* $1/M_P$ on lattice D, where M_P is the mass of a heavy-light pseudoscalar meson (with light quark mass $\approx m_u, m_d$), and f_P is its decay constant. Two different interpolations to the B and D masses are shown.

One estimate of the systematic error of this approach is obtained at fixed lattice spacing by comparing two different mass ranges of propagating quarks: "lighter heavies," which give meson masses in the range 1.25 to 2 GeV, and "heavier heavies," which give meson masses in the range 2 to 4 GeV. Figure 2(b) shows the interpolations between the static result and the two propagating mass ranges. Other estimates of the systematic are available after one takes the $a \to 0$ limit — see below.

Perturbation Theory

The axial current A_μ with a lattice cutoff is not the same as the continuum A_μ, but the difference is perturbatively calculable, since it comes from physics at the cutoff scale. At present, we use a (mass independent) one loop matching, with the scale ("q^*") of the coupling estimated along the lines of Ref. [7]. For propagating Wilson quarks, the result is, after tadpole improvement, $q^* = 2.32/a$ [8]. We estimate the systematic error by changing q^* by a factor of 2 and reanalyzing. The error is rather small ($\lesssim 3\%$).

Finite Volume

We compare results on lattice A ($\ell = 1.2$ fm) with those on lattice B ($\ell = 2.5$ fm). All other systematics on these two lattices are the same. A is smaller than all other lattices used for our central value; B, much larger. Therefore the difference should give a conservative bound on the finite volume error. This error is $\sim$3–4% on decay constants and $\sim$1–2% on ratios.

Extrapolation to the Continuum ($a \to 0$)

For any physical quantity Q computed here, we expect (Wilson fermions)

$$Q(a) = Q_{a=0}(1 + a\mathcal{M}_1 + a^2\mathcal{M}_2 + \cdots) \ , \tag{3}$$

but we do not know $\mathcal{M}_1$ *a priori*, nor how it compares to $\mathcal{M}_2$. In practice, we find the slope to be quite large for the decay constants ($\mathcal{M}_1 \sim$ 300–650 MeV), with f_{B_s} the worst offender. This leads to rather large extrapolation errors ($\sim$ 12–27%). The ratios of decay constant are much better behaved, with $\mathcal{M}_1 \sim 100$ MeV, and an error of $\sim$4–5%.

Figure 3(a) shows several fits of f_B *vs.* a used to compute the central value and estimate the two largest sources of systematic error. The central value is obtained from a linear fit to the diamonds, which in turn use linear chiral fits, a lattice scale set by f_π, and the "EKM" corrections described above. The error of the continuum extrapolation is estimated by comparing the central value with the result of (1) a constant fit to the three diamonds with smallest values of a, (2) a linear fit to the squares, which use a different mass shift (the magnetic — m_3 — mass minus the pole mass) in the EKM corrections, and (3) a linear fit to the octagons, which use the "2κ" norm — *i.e.*, no EKM corrections. The continuum extrapolation error is defined as the largest of these three differences. The difference of the extrapolation of the crosses (which have a quadratic chiral extrapolation) and the central value determines the chiral extrapolation error.

Effects of Quenching

The quenched approximation has one great advantage: it saves an enormous amount of computer time. However, it is not a true approximation, since there is no perturbative expansion of the full theory for which the quenched approximation is the first term. Thus one should think of it only as a model. It is a good model since it has confinement and chiral symmetry breaking built in, is completely relativistic, seems to get the low lying hadronic spectrum right to ~ 5 to 10%, and can be shown to have small errors in a few cases [9]. But it is a model, nonetheless. Therefore one must try to estimate its errors, and ultimately to move beyond it.

To this end we have repeated our computations on lattices with virtual quark loops included. However, we emphasize that such computations are not yet "full QCD." This is because (1) the virtual quark mass is fixed and not extrapolated to physical up or down mass (the theory is partially quenched [4]), (2) the virtual quark data is not yet good enough to extrapolate to $a = 0$,

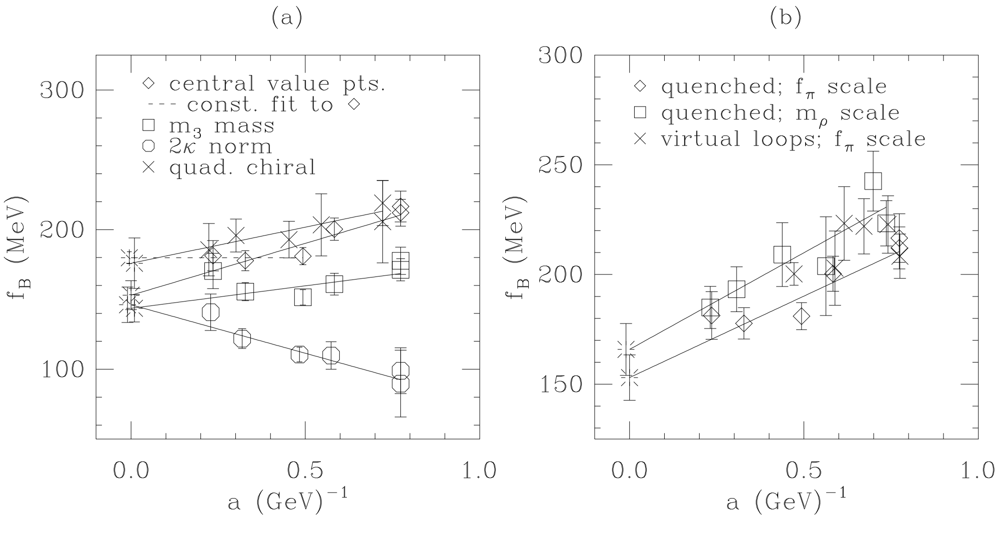

FIGURE 3. (a) Quenched results for f_B as a function of lattice spacing. The linear fit to all the diamonds gives the central value; other points and fits are used for estimating systematic errors — see text. (b) Results for f_B as a function of lattice spacing used for estimating the effects of quenching. The diamonds are the same as in (a); while the squares have the scale fixed by the mass of the ρ meson. The crosses (virtual quark loops included) are not extrapolated to the continuum.

and (3) we have two light flavors, not three. Thus the virtual quark simulations are used at this point only for systematic error estimation.

Figure 3(b) shows how the quenching error is estimated. One estimate is obtained by comparing the smallest-a virtual quark simulation (lattice G, the cross at $a = 0.47$ (GeV)$^{-1}$) with the quenched simulations, interpolated to the same value of a. Another estimate can be found by fixing the lattice scale in the quenched simulations by using m_ρ, instead of f_π. In principle, both methods of setting the scale should be just as good. However, quenching can affect m_ρ differently from f_π, so the difference is an estimate of the systematic.

We emphasize that our quenching error estimate is quite crude at present. The difference between our virtual quark simulations and full QCD must be kept in mind. Further, the comparison of f_π and m_ρ scale results tests the quenched approximation only under the assumption that other systematic errors are well controlled. The errors of our continuum extrapolation are large enough to make this last assumption rather shaky.

RESULTS

Currently, we find:

$$\begin{array}{llll}
f_{B_s} = 164 \ \pm 9 \ {}^{+47}_{-13} \ {}^{+16}_{-0} \ \text{MeV} & & f_{B_s}/f_B = 1.10 \ \pm 0.02 \ {}^{+0.05}_{-0.03} \ {}^{+0.03}_{-0.02} \\
f_B = 153 \ \pm 10 \ {}^{+36}_{-13} \ {}^{+13}_{-0} \ \text{MeV} & & f_{D_s}/f_D = 1.09 \ \pm 0.02 \ {}^{+0.05}_{-0.01} \ {}^{+0.02}_{-0.0} \\
f_{D_s} = 199 \ \pm 8 \ {}^{+40}_{-11} \ {}^{+10}_{-0} \ \text{MeV} & & f_{B_s}/f_{D_s} = 0.83 \ \pm 0.02 \ {}^{+0.06}_{-0.03} \ {}^{+0.03}_{-0.00} \\
f_D = 186 \ \pm 10 \ {}^{+27}_{-18} \ {}^{+9}_{-0} \ \text{MeV} & & f_B/f_{D_s} = 0.76 \ \pm 0.03 \ {}^{+0.07}_{-0.04} \ {}^{+0.02}_{-0.01}
\end{array} \quad (4)$$

The errors shown are statistical (plus "fitting"), systematic (within the quenched approximation), and systematic (of quenching), respectively. We note that as experimental measurements of f_{D_s} improve, the ratios f_B/f_{D_s} and f_{B_s}/f_{D_s} may ultimately provide the best way to determine f_B and f_{B_s}.

The most important issue still under study is whether to switch from the current linear chiral fits to quadratic ones in determining the central values. (The difference will still of course be counted as a systematic error.) If such a switch were to be made now, it would raise the central values of f_B, f_{B_s}, f_D and f_{D_s} by 23, 19, 13, and 14 MeV, respectively. The systematic error within the quenched approximation would then become much more symmetric, with the continuum extrapolation the dominant positive error and the chiral extrapolation the dominant negative one.

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