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Semileptonic Kaon Decay in Staggered Chiral Perturbation Theory

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Abstract

The determination of $|V_{us}|$ from kaon semileptonic decays requires the value of the form factor $f_+(q^2 = 0)$, which can be calculated precisely on the lattice. We provide the one-loop partially quenched staggered chiral perturbation theory expressions that may be employed to analyze staggered simulations of $f_+(q^2)$ with three light flavors. We consider both the case of a mixed action, where the valence and sea sectors have different staggered actions, and the standard case where these actions are the same. The momentum transfer q^2 of the form factor is allowed to have an arbitrary value. We give results for the generic situation where the u, d, and s quark masses are all different, $N_f = 1 + 1 + 1$, and for the isospin limit, $N_f = 2 + 1$. The expression we obtain for $f_+(q^2)$ is independent of the mass of the (valence) spectator quark. In the limit of vanishing lattice spacing, our results reduce to the one-loop continuum partially quenched expression for $f_+(q^2)$, which has not previously been reported in the literature for the $N_f = 1 + 1 + 1$ case. Our expressions have already been used in staggered lattice analyses of $f_+(0)$, and should prove useful in future calculations as well.

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I. INTRODUCTION

Elements of the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix are fundamental parameters of the weak interactions. In the Standard Model (SM) of particle physics, the matrix is unitary, so any violation of unitarity would point to new physical phenomena beyond the SM. No evidence of such new physics (NP) has yet been observed, but precision tests of unitarity provide stringent constraints on the allowed non-standard phenomena and the scale at which they may occur [1]. In particular, tests of the first row of the CKM matrix provide bounds in the scale of the NP that can contribute to these processes at the same level as those from Z-pole measurements [2].

The precision that can be achieved in these tests of the first row depend on the uncertainty in the determination of $|V_{ud}|$ and $|V_{us}|$, since $|V_{ub}|$ is negligible at the current level of precision. $|V_{ud}|$ is extracted from nuclear β decays [3], while the most precise determinations of $|V_{us}|$ come from kaon leptonic and semileptonic decays. Extracting $|V_{us}|$ from hadronic τ decays have the potential of being competitive with the above determinations [4], but they are currently limited by uncertainties in the experimental data [5]. Determinations of $|V_{us}|$ from kaon leptonic and semileptonic decays require non-perturbative inputs calculated on the lattice: the ratio of decay constants f_K/f_π [6–12] and the vector form factor $f_+(q^2=0)$ [13–19], respectively.

The vector form factor $f_+(q^2)$ is defined from the hadronic matrix element of a vector current for $K \to \pi l \nu$ processes

$$\langle \pi(p_\pi)|V^\mu|K(p_K)\rangle = f_+(q^2)\left[p_K^\mu + p_\pi^\mu - \frac{m_K^2 - m_\pi^2}{q^2}q^\mu\right] + f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2}q^\mu\,, \qquad (1.1)$$

where $q = p_K - p_\pi$ is the momentum transfer and $V^\mu = \bar{s}\gamma^\mu u$ is the appropriate flavor changing vector current. The CKM matrix element $|V_{us}|$ can thus be extracted using the form factor at zero momentum transfer, $f_+(0)$, experimental data for $\Gamma_{K_{l3(\gamma)}}$, and the relation [2]

$$\Gamma_{K_{l3(\gamma)}} = \frac{G_F^2 M_K^5 C_K^2}{128\pi^3} S_{\rm EW}|V_{us} f_+^{K^0\pi^-}(0)|^2 I_{Kl}^{(0)}\left(1 + \delta_{\rm EM}^{Kl} + \delta_{\rm SU(2)}^{K\pi}\right)\,, \qquad (1.2)$$

where the Clebsch-Gordon coefficient C_K is equal to 1 or $1/\sqrt{2}$ for neutral and charged kaons, respectively; $S_{\rm EW} = 1.0223(5)$ is the short-distance universal electroweak correction; and, $I_{Kl}^{(0)}$ is a phase space integral which depends on the shape of the form factors $f_\pm$. The parameters $\delta_{\rm EM}^{Kl}$ and $\delta_{\rm SU(2)}^{K\pi}$ contain long-distance electromagnetic and strong isospin-breaking corrections, respectively [2].

The error in $f_+(0)$ from lattice-QCD is now small enough so the uncertainty from $|V_{us}|$ from semileptonic decays in the unitarity test is comparable to that from $|V_{ud}|$ [19]. Further improvement in the calculation of $f_+(0)$ is however necessary in order to reach the same level of precision as the experimental input, $\Gamma_{K_{l3(\gamma)}}$. A key element for the reduction of the error in the state-of-the-art calculation of $f_+(0)$ [19] and previous lattice calculations using staggered fermions [16], as well as for future improvements planed by the Fermilab Lattice/MILC Collaboration, is the use of staggered chiral perturbation theory (SχPT) to analyze both the chiral behaviour and discretization corrections of the form factor.

Two different types of staggered simulations have been and are being used to determine $f_+(0)$ on the lattice. In Ref. [16], a "mixed-action" setup is used: the valence quarks have the

HISQ action [20], while the sea quarks have the asqtad action [21]. On the other hand, the second-generation calculation of $f_+(0)$ by the Fermilab Lattice/MILC Collaboration [18, 19] uses valence HISQ quarks on HISQ sea-quark ensembles generated by the MILC Collaboration [22], so there is no mismatch between sea and valence actions: the action is "unmixed."

In this paper we calculate $f_+(q^2)$ in SχPT for both situations. The chiral theory in the unmixed case is standard, so we simply review the formulation and notation, and then proceed to the calculation of $f_+(q^2)$. The mixed-action case, however, requires some modifications to the corresponding chiral theory, so we work those out first. It is then straightforward to modify the results for the form factor in the unmixed case to take into account the complications due to the mixed action.

Staggered quarks have a four-fold multiplicity of "taste" degrees of freedom, which result from the fermion doubling in the discretization of the Dirac equation. In staggered simulations, the unwanted tastes are removed by taking the fourth root of the quark determinant. At nonzero lattice spacing, the rooted theory then suffers from nonlocal violations of unitarity [23, 24]. However, theoretical arguments [25–28], as well as other analytical and numerical evidence [21, 29–32], indicate that standard QCD, both local and unitary, is recovered in the continuum limit. In the chiral theory, taking rooting into account is straightforward: each sea quark loop needs to be multiplied by a factor of 1/4 [33]. This can be accomplished systematically by replicating the sea quarks n_r times and taking $n_r = 1/4$ in the result of the chiral calculation [26, 28]. However, it is often easier to use the quark flow approach [34] to locate the loops, and then to simply insert the factors of 1/4 by hand. Since the quark-flow approach is also useful for other reasons in our calculations, we will in general use that method below.

This paper is organized as follows. General features of the form factors for $K_{\ell 3}$ decay, including the Ademollo-Gatto theorem [35], are discussed in Sec. II. In Sec. III we review the basics of SχPT. Section IV then discusses some details of the chiral perturbation theory for the mixed-action case, in which the valence and sea actions are different versions of staggered quarks, *e.g.*, HISQ and asqtad, respectively. The one-loop chiral calculation of the form factor $f_+(q^2)$ is performed in Sec. V. Although we consider arbitrary values of the valence quark masses, the result turns out to be independent of the mass of the valence spectator quark. This is a special property of $f_+(q^2)$, which must satisfy the Ademollo-Gatto theorem when the active (nonspectator) valence quarks are degenerate, and would not be true of the form factor $f_-(q^2)$. Both form factors do depend on the masses of all the quarks in the sea, which must enter symmetrically. The corresponding mixed-action results are presented in Sec. VI. We discuss our results and conclusions in Sec. VII. Appendix A introduces the needed one-loop momentum integrals and their evaluations, while Appendix B collects formulas in the special case of exact isospin in the sea (the 2+1 case, $m_u = m_d$) and in the continuum.

II. FORM FACTORS FOR $K_{\ell 3}$ DECAY

The hadronic matrix element between a kaon and a pion of the weak vector current may be parameterized by two form factors, f_+ and f_-, defined by

$$\langle \pi(p_\pi)|V^\mu|K(p_K)\rangle = f_+(q^2)\,(p_K^\mu + p_\pi^\mu) + f_-(q^2)\,(p_K^\mu - p_\pi^\mu) \ , \tag{2.1}$$

where $q = p_K - p_\pi$ is the momentum transfer, and $V^\mu = \bar{s}\gamma^\mu u$ is the appropriate flavor-changing vector current. It is often more convenient to introduce the scalar form factor f_0, defined by

$$f_0(q^2) = f_+(q^2) + f_-(q^2)\frac{q^2}{m_K^2 - m_\pi^2} \ , \tag{2.2}$$

in terms of which the matrix element is given by

$$\langle \pi(p_\pi)|V^\mu|K(p_K)\rangle = f_+(q^2)\left[p_K^\mu + p_\pi^\mu - \frac{m_K^2 - m_\pi^2}{q^2}q^\mu\right] + f_0(q^2)\frac{m_K^2 - m_\pi^2}{q^2}q^\mu \ . \tag{2.3}$$

This is useful phenomenologically because it is easier to disentangle f_+ and f_0 experimentally since they are less correlated than f_+ and f_-. In practice, the key quantity to be calculated on the lattice is the absolute normalization of the form factor f_+ at one value of q^2, which is usually taken to be $q^2 = 0$. Experiments provide the relative normalization $f_+(q^2)/f_+(0)$, so once $f_+(0)$ is known, the CKM element $|V_{us}|$ may be extracted from the total $K_{\ell 3}$ decay width. The kinematical relation $f_+(0) = f_0(0)$, which follows from Eq. (2.2), can be helpful in this regard [16].

In this paper we focus on a calculation of f_+ in SχPT. Although the main motivation is to aid in the lattice determination at $q^2 = 0$, we compute f_+ at arbitrary q^2, since this introduces few additional complications. For definiteness, we consider the mode $K^0 \to \pi^-\ell^+\nu$. That mode (and its charge conjugate) are the best-measured isospin channels experimentally, and other modes are usually normalized to it. For the purposes of the calculation here, the particular isospin mode considered is irrelevant, since we do not include electromagnetism, and we do a partially quenched calculation, so that the valence and sea masses may be chosen arbitrarily. For convenience for lattice computations, which usually are performed in the limit of exact isospin in the sea, we also provide results in that limit. In any case, the effect of isospin violation in the sea is NNLO — higher order than we are considering here.

The form factor $f_+(0)$ can be written as a χPT expansion:

$$f_+(0) = 1 + f_2 + f_4 + f_6 + ... = 1 + f_2 + \Delta f \ , \tag{2.4}$$

where the f_i contain corrections of $O(p^{2i})$ in the chiral power counting. The Ademollo-Gatto (AG) theorem [35], which follows from vector current conservation, ensures that $f_+(0) \to 1$ in the $SU(3)$ limit and, furthermore, that the $SU(3)$ breaking effects are second order in $(m_K^2 - m_\pi^2)$. This fixes f_2 in the continuum completely in terms of experimental quantities. The AG theorem is a statement about the valence quark masses in the mesons that enter the weak current (the valence u and s), so it remains true as a statement about the dependence on valence quark masses even in a partially quenched theory [36]. It is straightforward to see this using the approach to the AG theorem developed in Ref. [37], and it follows simply from the "U-spin" subgroup of (valence) $SU(3)$ symmetry that rotates the valence u and s into each other. Note that although Ref. [36] takes the spectator valence quark (the valence d) to be degenerate with the valence u, that is not necessary for the theorem to be valid, and we work below with arbitrary values of the three valence masses. Furthermore, since the theorem just depends on a flavor symmetry, it remains valid when staggered discretization effects are included through SχPT. We will verify below that the result of our calculation obeys the theorem. Nevertheless, violations of the AG theorem may be introduced at a later stage in a lattice computation. In particular, Ref. [16] uses the continuum dispersion

relation to relate the matrix element of the scalar density to that of the vector current of interest, and the dispersion relation is of course violated on the lattice. The corresponding discretization errors in the AG theorem appear to be very small, however [16].

III. BASICS OF STAGGERED CHIRAL PERTURBATION THEORY

Here, we follow the discussion in Ref. [21] fairly closely. The starting point for SχPT is the (Euclidean space) Lee-Sharpe Lagrangian [38] generalized to multiple flavors in Ref. [33]:

$$\mathcal{L} = \frac{f^2}{8}\mathrm{Tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) - \frac{1}{4}\mu f^2\mathrm{Tr}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) + \frac{m_0^2}{24}(\mathrm{Tr}(\Phi))^2 + a^2\mathcal{V} \ , \tag{3.1}$$

where the meson field Φ, $\Sigma \equiv \exp(i\Phi/f)$, and the quark mass matrix $\mathcal{M}$ are $4N_f \times 4N_f$ matrices, f is the pion decay constant at LO, and μ is a low energy constant (LEC). The parameter a is the lattice spacing, and discretization effects enter first at $O(a^2)$. We will assume that there are three light sea-quark flavors (u, d, and s), but N_f in general will be larger than three to accommodate valence quarks (and either additional ghost quarks or replicas), in order to allow for partial quenching [39, 40].

The field Σ transforms under $\mathrm{SU}(4N_f)_L\times\mathrm{SU}(4N_f)_R$ as $\Sigma \to L\Sigma R^\dagger$. The field Φ is given by:

$$\Phi = \begin{pmatrix} U & \pi^+ & K^+ & \cdots \\ \pi^- & D & K^0 & \cdots \\ K^- & \bar{K}^0 & S & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.2}$$

where each entry is a 4×4 matrix in taste space, with, for example,

$$\pi^+ \equiv \sum_{\Xi=1}^{16} \pi^+_\Xi T_\Xi. \tag{3.3}$$

The 16 Hermitian taste generators T_Ξ are

$$T_\Xi \in \{\xi_5, i\xi_{\mu 5}, i\xi_{\mu\nu}(\mu > \nu), \xi_\mu, I\}. \tag{3.4}$$

Here we use the Euclidean gamma matrices ξ_μ, with $\xi_{\mu\nu} \equiv \xi_\mu\xi_\nu$ ($\mu < \nu$), $\xi_{\mu 5} \equiv \xi_\mu\xi_5$, and $\xi_I \equiv I$ is the 4×4 identity matrix. The mass matrix has the form

$$\mathcal{M} = \begin{pmatrix} m_u I & 0 & 0 & \cdots \\ 0 & m_d I & 0 & \cdots \\ 0 & 0 & m_s I & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{3.5}$$

For generality, we will usually take all sea masses nondegenerate (*i.e.*, the 1+1+1 case) below. Converting the formulae to the 2+1 ($m_u = m_d$) case, which is relevant for current simulations [21, 22], is straightforward. One-loop results for $f_+(q^2)$ in the 2+1 case and in the continuum are compiled in Appendix B.

The quantity m_0 in Eq. (3.1) is the anomaly contribution to the mass of the singlet-taste and singlet-flavor meson, the $\eta' \propto \mathrm{Tr}(\Phi)$. As usual, the η' decouples in the limit $m_0 \to \infty$. However, one may postpone taking the limit and keep the η' as a dynamical field [41] in order to avoid putting conditions on the diagonal elements of Φ. The diagonal fields, $U, D, \ldots$, are then simply the $u\bar{u}$, $d\bar{d}$, ... bound states, which makes it easy to follow the quark flow through chiral diagrams [34] by following the flavor indices. A quark flow analysis is particularly useful in our calculations here because it helps keep track of the many different possible contributions that all correspond to the same diagram at the chiral (meson) level.

The taste-violating potential $\mathcal{V}$ in Eq. (3.1) is given by

$$\begin{aligned}
-\mathcal{V} \;=\; & C_1 \mathrm{Tr}(\xi_5^{(\mathrm{N_f})}\Sigma\xi_5^{(\mathrm{N_f})}\Sigma^\dagger) + \frac{\mathrm{C}_3}{2}[\mathrm{Tr}(\xi_\nu^{(\mathrm{N_f})}\Sigma\xi_\nu^{(\mathrm{N_f})}\Sigma) + \mathrm{h.c.}] \\
& + \frac{C_4}{2}[\mathrm{Tr}(\xi_{\nu 5}^{(\mathrm{N_f})}\Sigma\xi_{5\nu}^{(\mathrm{N_f})}\Sigma) + \mathrm{h.c.}] + \frac{\mathrm{C}_6}{2}\,\mathrm{Tr}(\xi_{\mu\nu}^{(\mathrm{N_f})}\Sigma\xi_{\nu\mu}^{(\mathrm{N_f})}\Sigma^\dagger) \\
& + \frac{C_{2V}}{4}[\mathrm{Tr}(\xi_\nu^{(\mathrm{N_f})}\Sigma)\mathrm{Tr}(\xi_\nu^{(\mathrm{N_f})}\Sigma) + \mathrm{h.c.}] + \frac{\mathrm{C}_{2\mathrm{A}}}{4}[\mathrm{Tr}(\xi_{\nu 5}^{(\mathrm{N_f})}\Sigma)\mathrm{Tr}(\xi_{5\nu}^{(\mathrm{N_f})}\Sigma) + \mathrm{h.c.}] \\
& + \frac{C_{5V}}{2}[\mathrm{Tr}(\xi_\nu^{(\mathrm{N_f})}\Sigma)\mathrm{Tr}(\xi_\nu^{(\mathrm{N_f})}\Sigma^\dagger)] + \frac{\mathrm{C}_{5\mathrm{A}}}{2}[\mathrm{Tr}(\xi_{\nu 5}^{(\mathrm{N_f})}\Sigma)\mathrm{Tr}(\xi_{5\nu}^{(\mathrm{N_f})}\Sigma^\dagger)],
\end{aligned} \tag{3.6}$$

with implicit sums over repeated indices. The $\xi_b^{(N_f)}$ are block-diagonal $4N_f \times 4N_f$ matrices:

$$\xi_b^{(N_f)} = \begin{pmatrix} \xi_b & 0 & 0 & \cdots \\ 0 & \xi_b & 0 & \cdots \\ 0 & 0 & \xi_b & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \tag{3.7}$$

with ξ_b the 4×4 objects, and $b \in \{5, \mu, \mu\nu\ (\mu < \nu), \mu 5, I\}$.

The two-trace terms in $\mathcal{V}$ generate two-point ("hairpin") vertices at $\mathcal{O}(a^2)$ that mix flavor-neutral particles of vector and axial tastes. In addition, flavor-neutral, singlet-taste particles are mixed by the m_0^2 term in Eq. (3.1), which results from the anomaly. For taste Ξ, we thus have terms the Lagrangian of the form $(\delta_\Xi/2)(U_\Xi + D_\Xi + S_\Xi + \cdots)^2$, where[1]

$$\delta_\Xi = \begin{cases} a^2\delta_V \equiv 16a^2(C_{2V} - C_{5V})/f^2, & \Xi \in \{\xi_\mu\} \text{ (vector taste)}; \\ a^2\delta_A \equiv 16a^2(C_{2A} - C_{5A})/f^2, & \Xi \in \{\xi_5\xi_\mu\} \text{ (axial taste)}; \\ 4m_0^2/3, & \Xi = I \text{ (singlet taste)}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

These mixings require us to diagonalize the full mass matrix in each of the three nontrivial taste channels. We write the neutral propagator for taste Ξ as:

$$G_\Xi \;=\; G_{0,\Xi} + \mathcal{D}^\Xi \;, \tag{3.9}$$

where $\mathcal{D}^\Xi$ is the part of the flavor-neutral propagator of taste Ξ that is disconnected at the

[1] Note that for vector and axial tastes we use the notation $\delta_{A,V}$, instead of $\delta'_{A,V}$ used in Ref. [33], to avoid cluttering the notation in the mixed-action case.

quark level (plus all iterations of intermediate sea quark loops). Explicitly, the disconnected propagator for a valence meson X (made out of valence quarks x and $\bar{x}$) and Y (similarly composed of y and $\bar{y}$) is [33]:

$$\mathcal{D}^{\Xi}_{XY}(p) \;=\; -a^2\delta_\Xi \frac{(p^2+m^2_{U_\Xi})(p^2+m^2_{D_\Xi})(p^2+m^2_{S_\Xi})}{(p^2+m^2_{X_\Xi})(p^2+m^2_{Y_\Xi})(p^2+m^2_{\pi^0_\Xi})(p^2+m^2_{\eta_\Xi})(p^2+m^2_{\eta'_\Xi})} \;. \tag{3.10}$$

Here, $m^2_{\pi^0_\Xi}$, $m^2_{\eta_\Xi}$ and $m^2_{\eta'_\Xi}$ are the eigenvalues of the full mass-squared matrix of the neutral sea mesons. For the singlet-taste propagator, we may simplify the disconnected propagator by taking $m_0 \to \infty$, and using the fact that, after rooting, $m_{\eta'_I} \approx m_0$ for large m_0. We obtain

$$\mathcal{D}^{I}_{XY}(p) \;=\; -\frac{4}{3}\frac{(p^2+m^2_{U_I})(p^2+m^2_{D_I})(p^2+m^2_{S_I})}{(p^2+m^2_{X_I})(p^2+m^2_{Y_I})(p^2+m^2_{\pi^0_I})(p^2+m^2_{\eta_I})} \;. \tag{3.11}$$

For the disconnected propagators in the vector-taste and axial-taste case, there is no explicit difference in Eq. (3.10) between the rooted and unrooted cases, but the sea masses in the denominators are dependent on the number of sea-quark flavors coming from intermediate loops, and one should use the appropriate masses in each case [33]. Note that, when $m_u = m_d$, the factors of $p^2 + m^2_{\pi^0}$ in Eqs. (3.10) and (3.11) cancel the factors of $p^2 + m^2_D$.

At leading order in SχPT, the mass of a pseudoscalar meson ("pion") of taste Ξ made of quarks with flavor i, j is given by

$$M^2_{ij,\Xi} = \mu(m_i + m_j) + a^2\Delta_\Xi \,, \tag{3.12}$$

where μ is the LEC in Eq. (3.1), and Δ_Ξ is the taste splitting, which can be written as a linear combination of the LECs C_1, C_3, C_4, and C_6 in Eq. (3.6). The splitting vanishes for the pseudoscalar-taste pion ($\Delta_5 = 0$), so it is a true Goldstone boson of the lattice theory in the chiral limit. The standard staggered power counting, which we follow here, assumes that the taste splittings and squared Goldstone pion masses are comparable. This is true not only for the MILC asqtad ensembles, but also for the MILC HISQ ensembles, which have smaller taste splittings but have smaller quark masses as well. Schematically, one describes the power counting by saying $a^2 \sim m$, where m is a generic quark mass.

IV. STAGGERED CHIRAL PERTURBATION THEORY WITH A MIXED STAGGERED ACTION

We now turn to the case of a mixed staggered theory, where the actions for the sea quarks and for the valence quarks are different, although both are staggered. We work out the staggered chiral Lagrangian for this case by starting with the quark-level Symanzik effective theory, which encodes the discretization errors as (higher dimensional) continuum operators. Once we have the Symanzik theory, it is straightforward to find the corresponding chiral theory, following a "spurion" analysis. This is the standard approach for including discretization errors in a chiral theory, first introduced by Sharpe and collaborators in Refs. [38, 42]. We note that, after we worked out the properties of staggered mixed-action chiral perturbation theory, we discovered Ref. [43], which developed mixed action staggered chiral perturbation theory several years ago, and found many of the results that are given

in this section.

A. Symanzik effective theory

Both the asqtad and the HISQ quarks have the full staggered set of symmetries. In particular they have separate $U(1)_\epsilon$ for each flavor, and there are overall rotation and shift symmetries. Rotations and shifts must be done on all staggered fields at once, since the the gluons must also be transformed. The analysis of the $\mathcal{O}(a^2)$ taste-violating four-quark operators in the Symanzik effective theory (SET) is then completely standard, and closely parallels the discussion in Ref. [21]. Recall that, in an ordinary (unmixed) staggered theory with n flavors, the four-quark operators are of the form

$$a^2 \bar{q}_i(\gamma_s \otimes \xi_t) q_i \; \bar{q}_j(\gamma_s \otimes \xi_t) q_j \ , \tag{4.1}$$

where i, j are (summed) flavor indices, and by $U(1)_\epsilon$ symmetry, the spin $\otimes$ taste combination $\gamma_s \otimes \xi_t$ must be odd with respect to $\gamma_5 \otimes \xi_5$. An example is tensor (T) $\otimes$ vector (V): $\gamma_{\mu\nu} \otimes \xi_\lambda$. In "type A" operators, the indices on the two γ_s matrices in Eq. (4.1) are the same, as are the indices on the two ξ_t matrices, but there are no indices in common between spin and taste. In "type B" operators some indices are repeated four times and are common to both spin and taste matrices. Type B operators break Euclidean invariance. They turn out to be irrelevant for the LO chiral theory because their chiral representatives require extra derivatives; they first appear at NLO [38].

In the mixed asqtad-HISQ theory, the operators are basically the same as above, but they come in three varieties: valence-valence, sea-sea, and sea-valence. Defining P_v and P_σ as projectors on the valence and sea quarks, we have, instead of Eq. (4.1):

$$\begin{aligned} & c_{vv}\, a^2\, \bar{q}(\gamma_s \otimes \xi_t) P_v q \; \bar{q}(\gamma_s \otimes \xi_t) P_v q \\ + \; & c_{\sigma\sigma}\, a^2\, \bar{q}(\gamma_s \otimes \xi_t) P_\sigma q \; \bar{q}(\gamma_s \otimes \xi_t) P_\sigma q \\ + \; & 2c_{v\sigma}\, a^2\, \bar{q}(\gamma_s \otimes \xi_t) P_v q \; \bar{q}(\gamma_s \otimes \xi_t) P_\sigma q \ , \end{aligned} \tag{4.2}$$

where flavor indices are now implied. The key point is that there are independent coefficients c_{vv}, $c_{\sigma\sigma}$, and $c_{v\sigma}$, because there is no lattice symmetry that turns valence and sea quarks into each other. The normalization in Eq. (4.2) is chosen so that the "unmixed limit" in which the valence and sea actions are identical is $c_{vv} = c_{\sigma\sigma} = c_{v\sigma}$. This follows from Eq. (4.1), if the flavor indices i, j are simply allowed to run over all quarks, both valence and sea.

B. Mixed theory chiral Lagrangian at leading order

The meson field ϕ is a matrix in flavor-taste space, where the flavor indices run over valence and sea indices.[2] We write

$$\phi \equiv \sum_{a=1}^{16} \phi^a T_a \ , \tag{4.3}$$

where ϕ^a is itself a matrix in flavor (really flavor-replica) space, and the Hermitian taste generators T_a are the same as in Eq. (3.4).

The chiral matrix Σ is defined as is in Eq. (3.1); it transforms under $SU(4n)_L \times SU(4n)_R$ as $\Sigma \to L\Sigma R^\dagger$, where n here denotes the total number of flavors (flavor-replicas) of all types. Valence quarks will be denoted by x, y, ..., and corresponding neutral valence mesons will be X, Y, ..., where the taste has not been specified. Similarly, sea quarks will be denoted u, d, s ..., and corresponding neutral valence mesons will be U, D, S ....

The spurion analysis to derive the chiral Lagrangian parallels the normal staggered case. Each taste spurion from Eq. (4.2) now comes with an additional projector (either P_v or P_σ) which leads to a 3-fold increase in the taste-violating chiral terms: They will be either vv, $\sigma\sigma$, or $v\sigma$, and their coefficients will be independent. To find these terms, we just have to take the normal taste-violating potential $\mathcal{V}$ [33] and insert appropriate projectors [3]. We will denote the corresponding taste-violating potentials as $\mathcal{V}_{vv}$, $\mathcal{V}_{\sigma\sigma}$ and $\mathcal{V}_{v\sigma}$. These potentials are given by:

$$\begin{aligned}
\mathcal{V}_{vv} &= \mathcal{U}_{vv} + \mathcal{U}'_{vv} \ ; \\
-\mathcal{U}_{vv} &= C_1^{vv}\ \mathrm{Tr}(\xi_5 P_v \Sigma \xi_5 P_v \Sigma^\dagger) + \frac{C_3^{vv}}{2}\ [\mathrm{Tr}(\xi_\mu P_v \Sigma \xi_\mu P_v \Sigma) + p.c.] \\
&\quad + \frac{C_4^{vv}}{2}\ [\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma \xi_{5\mu} P_v \Sigma) + p.c.] + \frac{C_6^{vv}}{2}\ \mathrm{Tr}(\xi_{\lambda\mu} P_v \Sigma \xi_{\mu\lambda} P_v \Sigma^\dagger) \\
-\mathcal{U}'_{vv} &= \frac{C_{2V}^{vv}}{4}\ [\mathrm{Tr}(\xi_\mu P_v \Sigma)\mathrm{Tr}(\xi_\mu P_v \Sigma) + p.c.] + \frac{C_{2A}^{vv}}{4}\ [\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma)\mathrm{Tr}(\xi_{5\mu} P_v \Sigma) + p.c.] \\
&\quad + \frac{C_{5V}^{vv}}{2}\ [\mathrm{Tr}(\xi_\mu P_v \Sigma)\mathrm{Tr}(\xi_\mu P_v \Sigma^\dagger)] + \frac{C_{5A}^{vv}}{2}\ [\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma)\mathrm{Tr}(\xi_{5\mu} P_v \Sigma^\dagger)] \ ,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\mathcal{V}_{\sigma\sigma} &= \mathcal{U}_{\sigma\sigma} + \mathcal{U}'_{\sigma\sigma} \ ; \\
-\mathcal{U}_{\sigma\sigma} &= C_1^{\sigma\sigma}\ \mathrm{Tr}(\xi_5 P_\sigma \Sigma \xi_5 P_\sigma \Sigma^\dagger) + \frac{C_3^{\sigma\sigma}}{2}\ [\mathrm{Tr}(\xi_\mu P_\sigma \Sigma \xi_\mu P_\sigma \Sigma) + p.c.] \\
&\quad + \frac{C_4^{\sigma\sigma}}{2}\ [\mathrm{Tr}(\xi_{\mu 5} P_\sigma \Sigma \xi_{5\mu} P_\sigma \Sigma) + p.c.] + \frac{C_6^{\sigma\sigma}}{2}\ \mathrm{Tr}(\xi_{\lambda\mu} P_\sigma \Sigma \xi_{\mu\lambda} P_\sigma \Sigma^\dagger) \\
-\mathcal{U}'_{\sigma\sigma} &= \frac{C_{2V}^{\sigma\sigma}}{4}\ [\mathrm{Tr}(\xi_\mu P_\sigma \Sigma)\mathrm{Tr}(\xi_\mu P_\sigma \Sigma) + p.c.] + \frac{C_{2A}^{\sigma\sigma}}{4}\ [\mathrm{Tr}(\xi_{\mu 5} P_\sigma \Sigma)\mathrm{Tr}(\xi_{5\mu} P_\sigma \Sigma) + p.c.] \\
&\quad + \frac{C_{5V}^{\sigma\sigma}}{2}\ [\mathrm{Tr}(\xi_\mu P_\sigma \Sigma)\mathrm{Tr}(\xi_\mu P_\sigma \Sigma^\dagger)] + \frac{C_{5A}^{\sigma\sigma}}{2}\ [\mathrm{Tr}(\xi_{\mu 5} P_\sigma \Sigma)\mathrm{Tr}(\xi_{5\mu} P_\sigma \Sigma^\dagger)] \ ,
\end{aligned} \tag{4.5}$$

[2] We will assume that we will use the quark flow or the replica method [40] to remove the valence-quark determinant; any needed replica indices in the latter case will be implicit.

[3] An exception occurs in the special case of singlet-taste operators, treated below.

$$
\begin{aligned}
\mathcal{V}_{v\sigma} &= \mathcal{U}_{v\sigma} + \mathcal{U}'_{v\sigma} \; ; \\
-\mathcal{U}_{v\sigma} &= C_1^{v\sigma} \left[\mathrm{Tr}(\xi_5 P_v \Sigma \xi_5 P_\sigma \Sigma^\dagger) + p.c.\right] + C_3^{v\sigma} \left[\mathrm{Tr}(\xi_\mu P_v \Sigma \xi_\mu P_\sigma \Sigma) + p.c.\right] \\
&\quad + C_4^{v\sigma} \left[\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma \xi_{5\mu} P_\sigma \Sigma) + p.c.\right] + \frac{C_6^{v\sigma}}{2} \left[\mathrm{Tr}(\xi_{\lambda\mu} P_v \Sigma \xi_{\mu\lambda} P_\sigma \Sigma^\dagger) + p.c.\right] \\
-\mathcal{U}'_{v\sigma} &= \frac{C_{2V}^{v\sigma}}{2} \left[\mathrm{Tr}(\xi_\mu P_v \Sigma)\mathrm{Tr}(\xi_\mu P_\sigma \Sigma) + p.c.\right] + \frac{C_{2A}^{v\sigma}}{2} \left[\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma)\mathrm{Tr}(\xi_{5\mu} P_\sigma \Sigma) + p.c.\right] \\
&\quad + \frac{C_{5V}^{v\sigma}}{2} \left[\mathrm{Tr}(\xi_\mu P_v \Sigma)\mathrm{Tr}(\xi_\mu P_\sigma \Sigma^\dagger) + p.c.\right] + \frac{C_{5A}^{v\sigma}}{2} \left[\mathrm{Tr}(\xi_{\mu 5} P_v \Sigma)\mathrm{Tr}(\xi_{5\mu} P_\sigma \Sigma^\dagger) + p.c.\right] \; . \quad (4.6)
\end{aligned}
$$

As usual [33], we denote the one-trace terms that contribute to tree-level mass splittings by $\mathcal{U}$, and the two-trace terms that give rise to the flavor-singlet taste-violating hairpins by $\mathcal{U}'$. The notation "p.c." implies the parity conjugate ($\Sigma \leftrightarrow \Sigma^\dagger$). It sometimes gives a different result from Hermitian conjugation (*e.g.*, in the $C_1^{v\sigma}$ terms, where it has the effect of switching $P_v \leftrightarrow P_\sigma$); the potentials are of course still Hermitian. Note that the size of each operator in $\mathcal{U}'_{v\sigma}$ has been "doubled" relative to the corresponding operators in $\mathcal{U}'_{vv}$ or $\mathcal{U}'_{\sigma\sigma}$; this is accomplished either by changing the overall coefficient or by adding the parity conjugate where it is needed. The normalization is convenient because then the unmixed limit where sea and valence actions are the same becomes $C_k^{vv} = C_k^{\sigma\sigma} = C_k^{v\sigma}$, where $k \in \{1, 3, 4, 6, 2V, 2A, 5V, 5A\}$.[4]

The case where $\xi_t = I$ in Eq. (4.2), *i.e.*, singlet-taste bilinears, is special. These operators don't contribute in a normal (unmixed) staggered theory because the taste spurion in Eq. (4.1) is the identity, so they only generate trivial (constant) chiral operators. However, for the mixed theory, the spurion is a projector (P_v or P_σ). Therefore these operators will generate chiral operators that have no counterpart in the unmixed case. Since $\xi_t = I$, the spin γ_s in Eq. (4.2) is V or A. Thus, the chiral operators will be similar to those generated by V×P or A×P. They give a chiral operator proportional to $\mathrm{Tr}(\xi_5 \Sigma \xi_5 \Sigma^\dagger)$ in the unmixed case, so here we just need to replace each ξ_5 by either P_v or P_σ. The result is

$$
C_0^{vv}\, \mathrm{Tr}(P_v \Sigma P_v \Sigma^\dagger) + C_0^{\sigma\sigma}\, \mathrm{Tr}(P_\sigma \Sigma P_\sigma \Sigma^\dagger) + C_0^{v\sigma} \left[\mathrm{Tr}(P_v \Sigma P_\sigma \Sigma^\dagger) + \mathrm{Tr}(P_\sigma \Sigma P_v \Sigma^\dagger)\right] \; , \quad (4.7)
$$

where the form of the $C_0^{v\sigma}$ term is required by parity invariance.

The operator in Eq. (4.7) can be simplified because various linear combinations of the terms are just constants. Defining $P_\pm \equiv P_\sigma \pm P_v$, and inserting $P_\sigma = \frac{1}{2}(P_+ + P_-)$, $P_v = \frac{1}{2}(P_+ - P_-)$ in Eq. (4.7), we see that any term involving P_+ reduces to a constant (independent of the chiral fields), because P_+ is the identity, and $\Sigma\Sigma^\dagger = I$. With the more standard notation $P_- \equiv \tau_3$ [44, 45], this operator becomes

$$
-\, C_{\mathrm{mix}} \mathrm{Tr}(\tau_3 \Sigma \tau_3 \Sigma^\dagger) \quad (4.8)
$$

where $C_{\mathrm{mix}} \equiv (2C_0^{v\sigma} - C_0^{\sigma\sigma} - C_0^{vv})/4$ is a measure of the mismatch between the sea and valence actions, and would vanish if there were a lattice symmetry interchanging sea and valence quarks.

The leading-order Euclidean Lagrangian, in analogy with the unmixed case given in

[4] This follows from the comments on normalization following Eq. (4.2).

Eq. (3.1), is then

$$\mathcal{L}_{LO} = \frac{f^2}{8}\mathrm{Tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) - \frac{1}{4}B\,f^2\,\mathrm{Tr}(\mathcal{M}\Sigma + \mathcal{M}\Sigma^\dagger) + \frac{2m_0^2}{3}\,\phi_I^2$$
$$-a^2 C_{\rm mix}\mathrm{Tr}(\tau_3\Sigma\tau_3\Sigma^\dagger) + a^2\mathcal{V}_{vv} + a^2\mathcal{V}_{\sigma\sigma} + a^2\mathcal{V}_{v\sigma}, \tag{4.9}$$

where, again, $\mathcal{M}$ is the quark mass matrix, μ is a LEC that relates quark and meson masses, and m_0 is the η' mass term from the anomaly. The η' field in Eq. (4.9) is defined by $\phi_I = X_I + Y_I + \cdots + U_I + D_I + S_I \cdots$, where the subscript I indicates the taste singlet.

As usual the potentials have "accidental" $SO(4)$ taste symmetry [38], so all mesons fall into $SO(4)$ representations with tastes P, A, T, V, and I. The valence-valence and sea-sea mesons with non-singlet flavor get standard mass splittings from $\mathcal{U}_{vv}$ and $\mathcal{U}_{\sigma\sigma}$, respectively. The splittings of the mixed (valence-sea) mesons come not only from the $C_{\rm mix}$ term and $\mathcal{U}_{v\sigma}$, however, but also from $\mathcal{U}_{vv}$ and $\mathcal{U}_{\sigma\sigma}$ [46]. Expanding to quadratic order, we find that a mixed meson with valence flavor q, sea flavor $\mathcal{S}$, and taste Ξ gets mass

$$M^2_{q\mathcal{S},\Xi} = \mu(m_q + m_{\mathcal{S}}) + a^2\Delta^{v\sigma}_\Xi, \tag{4.10}$$

where the splittings $\Delta^{v\sigma}_\Xi$ are given by:

$$\Delta^{v\sigma}(\xi_5) \equiv \Delta^{v\sigma}_P = \frac{4}{f^2}\Big[4C_{\rm mix} - 2C_1^{v\sigma} + C_1^{vv} + C_1^{\sigma\sigma} - 8C_3^{v\sigma} + 4C_3^{vv} + 4C_3^{\sigma\sigma}$$
$$-8C_4^{v\sigma} + 4C_4^{vv} + 4C_4^{\sigma\sigma} - 12C_6^{v\sigma} + 6C_6^{vv} + 6C_6^{\sigma\sigma}\Big]$$
$$\Delta^{v\sigma}(\xi_{\mu5}) \equiv \Delta^{v\sigma}_A = \frac{4}{f^2}\Big[4C_{\rm mix} + 2C_1^{v\sigma} + C_1^{vv} + C_1^{\sigma\sigma} + 4C_3^{v\sigma} + 4C_3^{vv} + 4C_3^{\sigma\sigma}$$
$$-4C_4^{v\sigma} + 4C_4^{vv} + 4C_4^{\sigma\sigma} + 6C_6^{vv} + 6C_6^{\sigma\sigma}\Big]$$
$$\Delta^{v\sigma}(\xi_{\mu\nu}) \equiv \Delta^{v\sigma}_T = \frac{4}{f^2}\Big[4C_{\rm mix} - 2C_1^{v\sigma} + C_1^{vv} + C_1^{\sigma\sigma} + 4C_3^{vv} + 4C_3^{\sigma\sigma}$$
$$+4C_4^{vv} + 4C_4^{\sigma\sigma} + 4C_6^{v\sigma} + 6C_6^{vv} + 6C_6^{\sigma\sigma}\Big]$$
$$\Delta^{v\sigma}(\xi_\mu) \equiv \Delta^{v\sigma}_V = \frac{4}{f^2}\Big[4C_{\rm mix} + 2C_1^{v\sigma} + C_1^{vv} + C_1^{\sigma\sigma} - 4C_3^{v\sigma} + 4C_3^{vv} + 4C_3^{\sigma\sigma}$$
$$+4C_4^{v\sigma} + 4C_4^{vv} + 4C_4^{\sigma\sigma} + 6C_6^{vv} + 6C_6^{\sigma\sigma}\Big]$$
$$\Delta^{v\sigma}(\xi_I) \equiv \Delta^{v\sigma}_I == \frac{4}{f^2}\Big[4C_{\rm mix} - 2C_1^{v\sigma} + C_1^{vv} + C_1^{\sigma\sigma} + 8C_3^{v\sigma} + 4C_3^{vv} + 4C_3^{\sigma\sigma}$$
$$+8C_4^{v\sigma} + 4C_4^{vv} + 4C_4^{\sigma\sigma} - 12C_6^{v\sigma} + 6C_6^{vv} + 6C_6^{\sigma\sigma}\Big]. \tag{4.11}$$

In contrast, the splittings for a valence-valence meson are given by

$$\begin{aligned}
\Delta^{vv}(\xi_5) &\equiv \Delta_P^{vv} = 0 \\
\Delta^{vv}(\xi_{\mu 5}) &\equiv \Delta_A^{vv} = \frac{16}{f^2}\left(C_1^{vv} + 3C_3^{vv} + C_4^{vv} + 3C_6^{vv}\right) \\
\Delta^{vv}(\xi_{\mu\nu}) &\equiv \Delta_T^{vv} = \frac{16}{f^2}\left(2C_3^{vv} + 2C_4^{vv} + 4C_6^{vv}\right) \\
\Delta^{vv}(\xi_{\mu}) &\equiv \Delta_V^{vv} = \frac{16}{f^2}\left(C_1^{vv} + C_3^{vv} + 3C_4^{vv} + 3C_6^{vv}\right) \\
\Delta^{vv}(\xi_{I}) &\equiv \Delta_I^{vv} = \frac{16}{f^2}\left(4C_3^{vv} + 4C_4^{vv}\right).
\end{aligned} \tag{4.12}$$

Sea-sea mesons obey equations identical to Eq. (4.12) but with $vv \to \sigma\sigma$ everywhere.

Note that in general a pseudoscalar-taste mixed meson is not a Goldstone boson because the required axial symmetry would interchange valence and sea quarks and is not a lattice symmetry in the mixed action case. Thus its mass has a non-zero contribution proportional to a^2 and independent of the quark masses, unlike the pseudoscalar-taste valence-valence or sea-sea mesons. For the mixed P meson, this contribution would vanish if $C_{\rm mix} = 0$ and $C_k^{v\sigma} = C_k^{vv} = C_k^{\sigma\sigma}$ for $k \in \{1, 3, 4, 6, 2V, 2A, 5V, 5A\}$; this would be required if sea and valence had the same action, but not otherwise. One can also check from Eqs. (4.11) and (4.12) that under these conditions all splittings for valence-valence, sea-sea, and valence-sea mesons are identical.

Upon expanding $\mathcal{U}'_{vv}$, $\mathcal{U}'_{v\sigma}$ and $\mathcal{U}'_{\sigma\sigma}$ in Eqs. (4.4), (4.5) and (4.6) to quadratic order, we find two-point vertices ("hairpins") coupling flavor-neutral mesons for both axial tastes and vector tastes. In particular, in the axial case, there are the following quadratic terms in $\mathcal{L}_{LO}$:

$$\tfrac{1}{2}a^2\delta_A^{vv}\,(X_{\mu 5} + Y_{\mu 5} + \cdots)^2\,; \qquad \delta_A^{vv} \equiv \frac{16}{f^2}(C_{2A}^{vv} - C_{5A}^{vv})\,, \tag{4.13}$$

$$\tfrac{1}{2}a^2\delta_A^{\sigma\sigma}\,(U_{\mu 5} + D_{\mu 5} + S_{\mu 5} + \cdots)^2\,; \qquad \delta_A^{\sigma\sigma} \equiv \frac{16}{f^2}(C_{2A}^{\sigma\sigma} - C_{5A}^{\sigma\sigma})\,, \tag{4.14}$$

$$a^2\delta_A^{v\sigma}\,(X_{\mu 5} + Y_{\mu 5} + \cdots)(U_{\mu 5} + D_{\mu 5} + S_{\mu 5} + \cdots)\,; \qquad \delta_A^{v\sigma} \equiv \frac{16}{f^2}(C_{2A}^{v\sigma} - C_{5A}^{v\sigma})\,. \tag{4.15}$$

Taking into account the minus sign from e^{-S}, this means that the two-point vertex coupling $X_{\mu 5}$ and $Y_{\mu 5}$ is $-\delta_A^{vv}$; that coupling $U_{\mu 5}$ and $D_{\mu 5}$ is $-\delta_A^{\sigma\sigma}$; and that coupling $X_{\mu 5}$ and $U_{\mu 5}$ is $-\delta_A^{v\sigma}$, *etc.* The vector-taste case is similar, with simply $A \to V$ and $\mu_5 \to \mu$.

There are also standard hairpins in the singlet-taste channel, coming from the anomaly (m_0^2) term in $\mathcal{L}_{LO}$. This produces a vertex of strength $-4m_0^2/3$ between all singlet-taste, flavor-neutral mesons, *e.g.*, between X_I and Y_I, between U_I and D_I, and between X_I and U_I. Note that, unlike the unphysical taste-violating hairpins above, there is no possibility of different strengths in the sea and valence sectors for this physical singlet-taste vertex, even after including $\mathcal{O}(a^2)$ corrections. The reason is that the (non-anomalous) continuum chiral symmetries require that the anomaly term can only be a function of the equally weighted sum $X_I + Y_I + \cdots + U_I + D_I + S_I + \cdots \propto \operatorname{tr}\ln(\Sigma) = \ln\det(\Sigma)$, not $X_I + Y_I + \cdots$ or $U_I + D_I + S_I + \cdots$ separately. At $\mathcal{O}(a^2)$, any new operators must be constructed from Σ, $\Sigma^\dagger$, and the spurions. Separate functions of $X_I + Y_I + \cdots$ or $U_I + D_I + S_{\mu 5} + \cdots$ are again

forbidden, basically because the determinant of a projector vanishes.[5]

The disconnected propagators in various taste channels can then be found by summing the geometric series with sea quark loop insertions, as usual. In the singlet-taste channel, this is completely standard, since the hairpin vertex is universal. The result for a disconnected propagator between valence meson X_I and valence meson Y_I has the same form as in the unmixed case, Eq. (3.11).

The only effect of the mixed action in Eq. (3.11) is that the splittings that contribute to the valence meson masses m_{X_I} and m_{Y_I} are different from the splittings contributing to the sea mesons m_{U_I}, m_{π_I}, *etc.* In fitting to chiral forms that result from mixed SχPT one should measure these splittings and input them: In the application in Ref. [16], they are just the valence HISQ splittings, and the normal asqtad sea splittings.

The axial-taste and vector-taste cases are more complicated because of the presence of three different hairpin coefficients in each channel, Eqs. (4.13), (4.14) and (4.15). In the series defining the disconnected vector-taste propagator, $D^V_{XY}(p^2)$, the first term, with no sea quark loops, is proportional to δ^{vv}_V (times two valence propagators) since the valence mesons couple to each other directly. The next term is proportional to $(\delta^{v\sigma}_V)^2$, since the valence mesons couple to the single sea quark loop. After that, each additional sea quark loop brings in a factor of $\delta^{\sigma\sigma}_V$, as the sea mesons couple to each other, in addition to the overall $(\delta^{v\sigma}_V)^2$ factor. Thus, all terms except the first form a geometric series that, aside from an overall factor of $(\delta^{v\sigma}_V/\delta^{\sigma\sigma}_V)^2$, is identical to the corresponding series in the pure sea (asqtad) theory. There is a mismatch in the first term, however, which would be proportional to $(\delta^{v\sigma}_V)^2/\delta^{\sigma\sigma}_V$ if the same correspondence held. Thus we need to add and subtract a term like the first one, but with a factor of $(\delta^{v\sigma}_V)^2/\delta^{\sigma\sigma}_V$ instead of δ^{vv}_V. The result is:

$$\begin{aligned}\mathcal{D}^V_{XY}(p) \;=\; & -\frac{a^2(\delta^{v\sigma}_V)^2/\delta^{\sigma\sigma}_V}{(p^2+m^2_{X_V})(p^2+m^2_{Y_V})}\,\frac{(p^2+m^2_{U_V})(p^2+m^2_{D_V})(p^2+m^2_{S_V})}{(p^2+m^2_{\pi_V})(p^2+m^2_{\eta_V})(p^2+m^2_{\eta'_V})} \\ & -\frac{a^2[\delta^{vv}_V-(\delta^{v\sigma}_V)^2/\delta^{\sigma\sigma}_V]}{(p^2+m^2_{X_V})(p^2+m^2_{Y_V})}\;. \end{aligned} \tag{4.16}$$

For the axial-taste channel, just let $V \to A$ everywhere.

Thus there is a "normal" hairpin term with strength $(\delta^{v\sigma}_\Xi)^2/\delta^{\sigma\sigma}_\Xi$ (with $\Xi = A, V$), plus an additional product of two poles proportional to $\delta^{vv}_\Xi - (\delta^{v\sigma}_\Xi)^2/\delta^{\sigma\sigma}_\Xi$. Note that this additional term means that there is a real double pole (for $m_{X_\Xi} = m_{Y_\Xi}$) even if the quark masses are tuned to the limit where valence-valence and sea-sea masses are equal in the taste-Ξ (or any other taste) channel.[6] This is an example of the sickness of a mixed-action theory; of course it goes away in the continuum limit.

A "factorization" assumption about the four-quark operators suggests a natural size for the mixed hairpin coefficients $\delta^{v\sigma}_\Xi$. Factorization is the assumption that the four quark operators take the form of squares of bilinears: $(c_v\bar{q}P_vq + c_\sigma\bar{q}P_\sigma q)^2$. One would then expect $\delta^{v\sigma}_\Xi \sim \sqrt{\delta^{vv}_\Xi\delta^{\sigma\sigma}_\Xi}$. Note that this only an order of magnitude argument; even if factorization were exact for every four-quark operator, $\delta^{v\sigma}_\Xi$ would not equal $\sqrt{\delta^{vv}_\Xi\delta^{\sigma\sigma}_\Xi}$ unless the ratio c_σ/c_v

[5] We thank M. Golterman for discussions on this issue.

[6] In practice, it is most common to tune valence-valence and sea-sea masses equal in the pseudoscalar-taste (Goldstone) channel.

were also identical for each operator contributing to the hairpins. With this guess for the size of the δ_Ξ parameters, we expect that the coefficient of the normal disconnected propagator is $\sim \delta_\Xi^{vv}$, *i.e.*, comparable to a typical HISQ taste-splitting. This estimate also suggests that the coefficient of the new double-pole term may be rather small. It is therefore convenient to define the parameter $\delta_\Xi^{\rm mix} = \delta_\Xi^{vv} - (\delta_\Xi^{v\sigma})^2/\delta_\Xi^{\sigma\sigma}$ and write the SχPT expressions in terms of $\delta_\Xi^{\rm mix}$ (which we expect to be suppressed with respect to the parameters in the sea and the valence sectors) and δ_Ξ^{vv}:

$$\begin{aligned}\mathcal{D}^{\Xi}_{XY}(p) \;=\; & -\frac{a^2(\delta_\Xi^{vv}-\delta_\Xi^{\rm mix})}{(p^2+m^2_{X_\Xi})(p^2+m^2_{Y_\Xi})}\frac{(p^2+m^2_{U_\Xi})(p^2+m^2_{D_\Xi})(p^2+m^2_{S_\Xi})}{(p^2+m^2_{\pi_\Xi})(p^2+m^2_{\eta_\Xi})(p^2+m^2_{\eta'_\Xi})} \\ & -\frac{a^2\delta_\Xi^{\rm mix}}{(p^2+m^2_{X_I})(p^2+m^2_{Y_I})}\;. \end{aligned} \tag{4.17}$$

We emphasize that Eq. (4.17) is for the vector-taste and axial-taste cases only; the mixed-action singlet-taste disconnected propagator has the identical form to the unmixed version, either Eq. (3.10) for $\Xi = I$, or Eq. (3.11) after rooting and $m_0 \to \infty$. Although available data is not enough to determine the value of δ_Ξ^{mix} precisely, current chiral fits to data using the asqtad action in the sea sector and the HISQ action in the valence sector [16] prefer non-zero values that are of the same sign but an order of magnitude smaller than δ_Ξ^{vv}.

The partially quenched one-loop calculation involves mesons made of two valence, two sea, and one valence and one sea quarks. The corresponding masses are given by the definition in Eq. (3.12) with the taste splittings Δ_Ξ being Δ_Ξ^{vv}, $\Delta_\Xi^{\sigma\sigma}$, and $\Delta_\Xi^{v\sigma}$, respectively.

We note that the taste-violating neutral propagator in the staggered mixed-action theory, Eq. (4.16), has been derived previously by Bae *et al.* [43], as have our estimates for the natural size of the new mixed-action hairpin parameters.

V. ONE-LOOP CALCULATION OF $f_+(q^2)$

Here we perform the chiral calculation of the form factor $f_+(q^2)$ at one-loop in standard SχPT, that is, with an unmixed staggered action. In Sec. VI, we give the results again (but not the detailed calculation) for the mixed-action case. The NLO contribution for $q^2 \neq 0$ is given by the sum of the one-loop contributions that we denote $f_2(q^2)$ in analogy to Eq. (2.4), plus an analytical term $\frac{4}{f^2}L_9^r(\mu)q^2$, where $L_9^r(\mu)$ is a renormalized NLO low energy constant and μ the scale at which the logarithms in $f_2(q^2)$ are evaluated. In the rest of the paper we omit this contribution in the analytical expressions but it should be added to our results for $f_2(q^2)$ for applications. We focus on the decay $K^0 \to \pi^-\ell^+\nu$, which, at the quark level, is due to the charged weak vector current $\bar{s}\gamma_\mu u$. Allowing for partial quenching, we let $\bar{y}$ be the valence antiquark corresponding to $\bar{s}$, $\bar{x}$ be the valence antiquark corresponding to $\bar{u}$, and x' be the spectator quark corresponding to d. Thus the decaying valence pseudoscalar is $x'\bar{y}$, the outgoing pseudoscalar is $x'\bar{x}$, and the current is $\bar{y}\gamma_\mu x$. The corresponding neutral valence mesons $x\bar{x}$, $x'\bar{x}'$, and $y\bar{y}$, are named X, X', and Y, respectively.

At the meson level, the vector current in continuum partially quenched chiral perturbation theory is

$$V^\mu_{xy} = \frac{if^2}{4}\left[\partial^\mu\Sigma\,\Sigma^\dagger - \Sigma^\dagger\partial^\mu\Sigma\right]_{xy}\;. \tag{5.1}$$

As always, the left index of Σ or $\Sigma^\dagger$ is a quark index, and the right index is an antiquark index.

For SχPT, we must choose the taste structure of the current. Since we want the current to be diagonal in both taste and flavor when $x = y$, in order that it be related to the quark number current and the Ademollo-Gatto theorem apply, we need to choose the singlet-taste current. Furthermore, a singlet-taste current will allow the incoming and outgoing mesons both to be pseudoscalar taste, which is by far the easiest choice for simulations. We thus take

$$V^{\mu}_{xy} = \frac{if^2}{4}\,\mathrm{tr_t}\big[\partial^\mu\Sigma\,\Sigma^\dagger - \Sigma^\dagger\partial^\mu\Sigma\big]_{xy}\ , \tag{5.2}$$

where $\mathrm{tr_t}$ is the trace over taste indices only.

We may check the normalization in Eq. (5.2) by computing the K–π matrix element at leading order (LO) in SχPT. In the unitary case ($y = s$, $x = u$, $x' = d$), the relevant two-meson term in the current is

$$i\sum_{\Xi}\left(\partial^\mu\pi^+_\Xi\,K^0_\Xi - \pi^+_\Xi\,\partial^\mu K^0_\Xi\right)\ , \tag{5.3}$$

with Ξ the taste of the mesons. The LO matrix element for pseudoscalar-taste mesons is then

$$\langle\pi^-_5(p_\pi)|\,V^{us}_\mu\,|K^0_5(p_K)\rangle = p^\mu_K + p^\mu_\pi\ , \tag{5.4}$$

where the subscript 5 indicates the pseudoscalar taste, that is $\Xi = \xi_5$. Thus, $f_+(q^2) = 1$ at LO, consistent with the Ademollo-Gatto theorem.

In the following subsections we collect the relevant one-loop SχPT formulae for the extrapolation of the form factor $f_+(q^2)$. We begin with the simplest contribution, wave function renormalization.

A. Wave function renormalization

The one-loop contribution to the wave function renormalization of a pseudoscalar meson P_{xy} (with pseudo scalar taste) can be taken from Ref. [33] or from Appendix A of Ref. [47]. We first write it in the compact notation using Eq. (A16), as well as Eqs. (A4) and (A6), for the chiral logarithm function ℓ:

$$\begin{aligned} Z_{P_{xy}} \;=\; \frac{1}{12(4\pi f)^2}\sum_{\Xi}\Bigg\{&\frac{1}{4}\sum_{\mathcal{S}}\left[\ell\left(m^2_{x\mathcal{S},\Xi}\right) + \ell\left(m^2_{y\mathcal{S},\Xi}\right)\right] \\ &+ \ell(D^{\Xi}_{XX}) + \ell(D^{\Xi}_{YY}) - 2c_\Xi\ell(D^{\Xi}_{XY})\Bigg\}\ , \end{aligned} \tag{5.5}$$

where Ξ runs over the 16 tastes, and $\mathcal{S}$ runs over the sea quarks (u, d, s). The disconnected propagators D^Ξ are given by Eq. (3.10) (or the simplified form of Eq. (3.11) for the singlet-taste channel). The rooting procedure, which multiplies the terms involving $\mathcal{S}$ by a factor of 1/4, and modifies the sea-meson masses entering into the denominators of the disconnected propagators, has already been implemented. The sign factor c_Ξ, which arises from commuting the pseudoscalar-taste external fields past the taste-Ξ internal fields, is defined by

$$c_\Xi = \frac{1}{4}\mathrm{Tr}\left(\xi_5\xi_\Xi\xi_5\xi_\Xi\right)\ . \tag{5.6}$$

Note that the taste of the external meson only enters through these factors; for an external taste other than pseudoscalar, one merely needs to replace each explicit ξ_5 matrix in Eq. (5.6) with the appropriate taste matrix.

We may also express the disconnected propagators as sums over simple poles times the residue functions $R_j^{[n,k]}$ defined in Ref. [33], and write the result in Eq. (5.5) more explicitly in terms of the chiral logarithm function $\ell(m^2)$, Eq. (A6), only. In the $N_f = 1+1+1$ case (*i.e.*, no degeneracies in the sea), we have

$$\begin{aligned}
Z_{P_{xy}} \;=\; & \frac{1}{3(4\pi f)^2}\Bigg\{\frac{1}{16}\sum_{\mathcal{S},\Xi}\left[\ell\left(m^2_{x\mathcal{S},\Xi}\right)+\ell\left(m^2_{y\mathcal{S},\Xi}\right)\right] \\
& + \frac{1}{3}\Bigg[\sum_{j\in\mathcal{M}^{(3,x)}}\frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \\
& + \sum_{j\in\mathcal{M}^{(3,y)}}\frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \\
& + 2\sum_{j\in\mathcal{M}^{(4,x,y)}}R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\Bigg] \\
& + a^2\delta_V\Bigg[\sum_{j\in\mathcal{M}^{(4,x)}}\frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \\
& + \sum_{j\in\mathcal{M}^{(4,y)}}\frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \\
& - 2\sum_{j\in\mathcal{M}^{(5,x,y)}}R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\Bigg] + \Big[V\to A\Big]\Bigg\}\,, \qquad (5.7)
\end{aligned}$$

Here the derivatives with respect to m_X^2 and m_Y^2 (of various tastes) arise from the double poles in the disconnected propagators D^{Ξ}_{XX} and D^{Ξ}_{YY}. The arguments $\mathcal{M}$ and μ of the residue functions $R_j^{[n,k]}$ are various sets of meson masses:

$$\begin{aligned}
\{\mathcal{M}_\Xi^{(3,z)}\} &\equiv \{m_{\pi^0,\Xi},\, m_{\eta,\Xi},\, m_{Z,\Xi}\}\,, \\
\{\mathcal{M}_\Xi^{(4,z,z')}\} &\equiv \{m_{\pi^0,\Xi},\, m_{\eta,\Xi},\, m_{Z,\Xi},\, m_{Z',\Xi}\}\,, \\
\{\mathcal{M}_\Xi^{(4,z)}\} &\equiv \{m_{\pi^0,\Xi},\, m_{\eta,\Xi},\, m_{\eta',\Xi},\, m_{Z,\Xi}\}\,, \\
\{\mathcal{M}_\Xi^{(5,z,z')}\} &\equiv \{m_{\pi^0,\Xi},\, m_{\eta,\Xi},\, m_{\eta',\Xi},\, m_{Z,\Xi},\, m_{Z',\Xi}\}\,, \qquad (5.8) \\
\{\mu_\Xi^{(3)}\} &\equiv \{m_{U,\Xi},\, m_{D,\Xi},\, m_{S,\Xi}\}\,,
\end{aligned}$$

where z and z' can be any valence quark (x, x' or y), and Z and Z' are the corresponding $z\bar{z}$ or $z'\bar{z}'$ mesons (X, X', or Y). For our partially quenched version of $K\to\pi l\nu$, we need $(Z_{P_{xx'}}+Z_{P_{yx'}})/2$.

In the 2+1 case, the π is degenerate with the diagonal U and D states, so π should be eliminated from the denominator sets $\mathcal{M}$, D (say) should be eliminated from the numerator

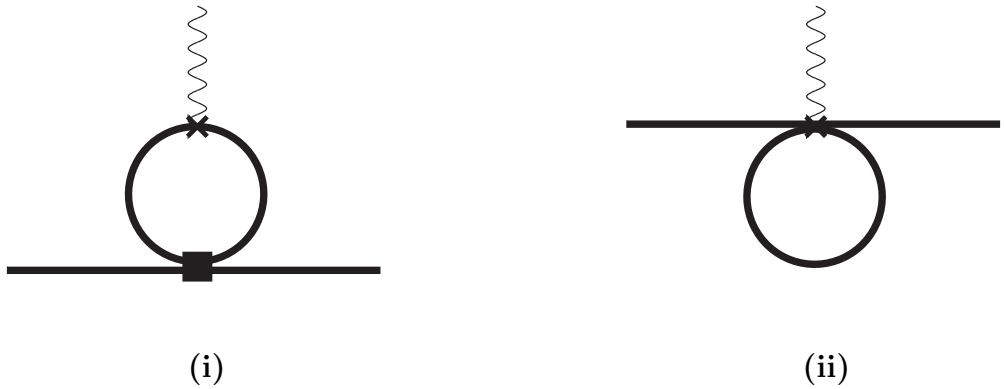


FIG. 1: Meson diagrams contributing to $f_+(q^2)$. The bold $\times$ with attached wavy line represents the weak vector current and the filled squared represents a strong vertex. Diagram (i) consists of the two-meson contribution to the current, together with a strong 4-meson vertex from the Lagrangian. Diagram (ii) has a four-meson contribution to the current.

set μ, and each index of each residue function should be reduced by 1:

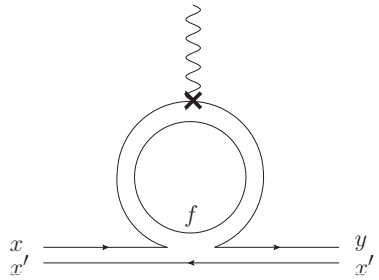
$$R_j^{[n,k]} \to R_j^{[n-1,k-1]} \ .$$

B. Strong vertex diagram

Aside from wave function renormalization, there are two chiral diagrams that contribute to the form factors $f_+(q^2)$ and $f_-(q^2)$, which are shown in Fig. 1. Diagram Fig. 1(i) contains an $O(p^0)$ $\Delta S = 1$ current vertex involving two fields and a strong $O(p^2)$ vertex involving four fields. We call it the "strong vertex" diagram, to distinguish it from Fig. 1(ii), which contains only a current vertex. A significant simplification to the evaluation of the strong vertex diagram comes from the fact that we calculate only the form factor f_+. The derivative ∂^μ in the current vertex introduces a factor of k^μ or $(k-q)^\mu$, where k is the loop momentum. Since the loop integrand depends only on k and q, in most cases the integration over k will give a result proportional to q^μ, so the diagram becomes a contribution to f_-, not f_+. For example, this will occur when the strong vertex in the graph comes from the mass term in the Lagrangian, Eq. (3.1). The only exception occurs when the vertex comes from the kinetic energy term, with one of its derivatives acting on an internal line and the other acting on an external line. This can introduce a factor of $k^\nu p^\nu$ where p is either p_K or p_π. The integration over k will then produce a term proportional to $\delta^{\mu\nu}$ (see Eq. (A3)). There is thus a contribution that goes like p^μ and hence can contribute to f_+ (as well as to f_-).

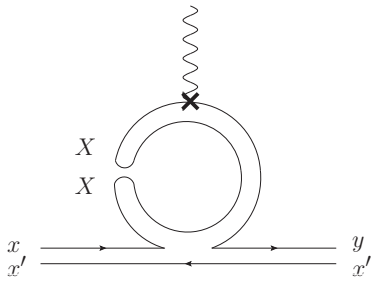
There are seven different topologies of this type. One, shown in Fig. 2, has all connected internal meson propagators, while six, shown in Fig. 3, have a disconnected meson propagator.

In the compact notation of Eq. (A17), as well as Eqs. (A5) and (A8), the strong vertex

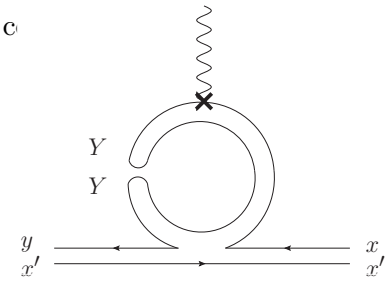


(a)

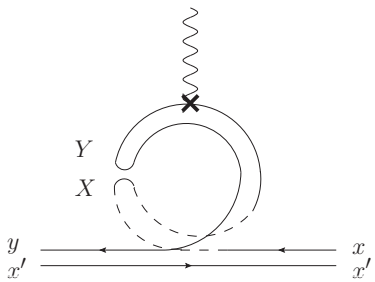
FIG. 2: ex diagram with c gators.



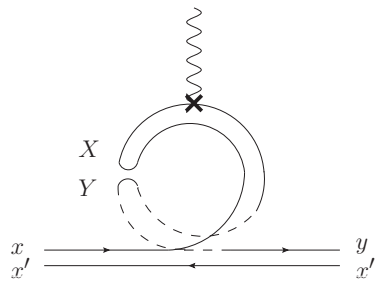
(b)



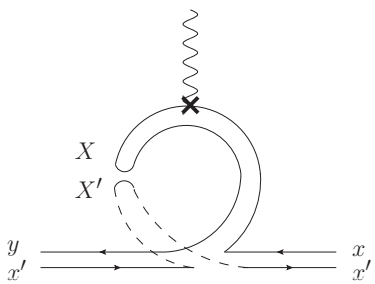
(c)



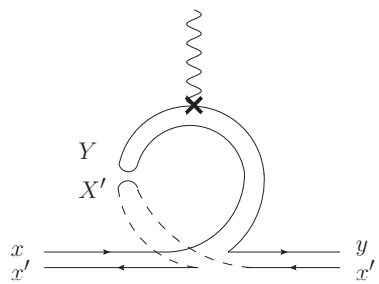
(d)



(e)



(f)



(g)

FIG. 3: Quark flow for the strong vertex diagrams.

diagrams give the following contribution to $f_+(q^2)$:

$$\frac{1}{2(4\pi f)^2}\sum_{\Xi}\Bigg\{-\frac{1}{4}\sum_{\mathcal{S}}\tilde{B}_{22}(m^2_{x\mathcal{S},\Xi},m^2_{y\mathcal{S},\Xi},q^2) \qquad (\mathbf{a})$$
$$-\tilde{B}_{22}(m^2_{xy,\Xi},D^{\Xi}_{XX},q^2) \qquad (\mathbf{b})$$
$$-\tilde{B}_{22}(m^2_{xy,\Xi},D^{\Xi}_{YY},q^2) \qquad (\mathbf{c})$$
$$+2\tilde{B}_{22}(m^2_{xy,\Xi},D^{\Xi}_{XY},q^2) \qquad (\mathbf{d})+(\mathbf{e})\Bigg\}, \tag{5.9}$$

where the bold letter(s) after each term indicate(s) the diagram(s) it comes from. Note that $\tilde{B}_{22}$ is a factor of $(4\pi)^2$ larger than the corresponding function $\bar{B}_{22}$ defined in Ref. [48]. Diagrams **f** and **g** do not contribute to $f_+(q^2)$ because the strong vertex gives no terms with the needed single factor of the loop momentum k; those diagrams do however contribute to $f_-(q^2)$. Since the only strong vertex diagrams where the spectator quark plays a role are **f** and **g**, Eq. (5.9) is independent of the spectator quark x'.

Explicitly, in the $N_f = 1+1+1$ case, Eq. (5.9) becomes

$$\frac{2}{(4\pi f)^2}\Bigg\{ \quad -\frac{1}{16}\sum_{\mathcal{S},\Xi}\tilde{B}_{22}(m^2_{x\mathcal{S},\Xi},m^2_{y\mathcal{S},\Xi},q^2) \quad (\mathbf{a})$$
$$+\frac{1}{3}\left[-\frac{\partial}{\partial m^2_{X,I}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R^{[3,3]}_j\left(\mathcal{M}^{(3,x)}_I;\mu^{(3)}_I\right)\right\}\right] \quad (\mathbf{b})$$
$$+\frac{1}{3}\left[-\frac{\partial}{\partial m^2_{Y,I}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R^{[3,3]}_j\left(\mathcal{M}^{(3,y)}_I;\mu^{(3)}_I\right)\right\}\right] \quad (\mathbf{c})$$
$$-\frac{2}{3}\left[\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R^{[4,3]}_j\left(\mathcal{M}^{(4,x,y)}_I;\mu^{(3)}_I\right)\right] \quad (\mathbf{d})+(\mathbf{e})$$
$$+a^2\delta_V\Bigg[\left(-\frac{\partial}{\partial m^2_{X,V}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R^{[4,3]}_j\left(\mathcal{M}^{(4,x)}_V;\mu^{(3)}_V\right)\right\}\right) \quad (\mathbf{b})$$
$$\left(-\frac{\partial}{\partial m^2_{Y,V}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R^{[4,3]}_j\left(\mathcal{M}^{(4,y)}_V;\mu^{(3)}_V\right)\right\}\right) \quad (\mathbf{c})$$
$$-2\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R^{[5,3]}_j\left(\mathcal{M}^{(5,x,y)}_V;\mu^{(3)}_V\right)\right) \quad (\mathbf{d})+(\mathbf{e})\Bigg]$$
$$+\Big[V\to A\Big]\Bigg\}. \tag{5.10}$$

where again the bold letter(s) after each term indicate(s) the diagram(s) it comes from. In Eq. (5.10) and below, the sums over j are always over all masses in the denominator mass sets $\mathcal{M}$ given in Eq. (5.8).

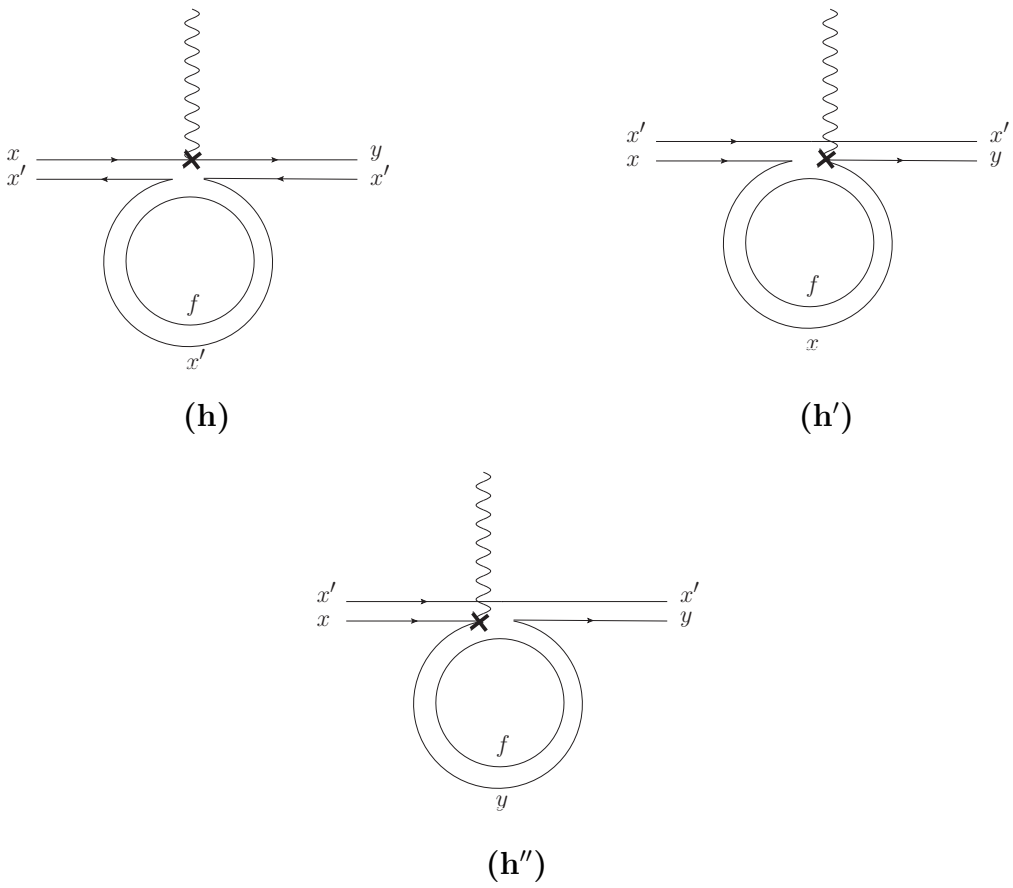


FIG. 4: Quark flow for current vertex diagrams that have connected internal meson propagators.

C. Current vertex diagram

In this type of diagram there is a $O(p^2)$ $\Delta S = 1$ vertex involving four fields. There are three topologies with connected internal mesons, shown in Fig. 4, and six with disconnected internal meson propagators, shown in Fig. 5.

In compact notation, the sum of the contribution from these diagrams to $f_+(q^2)$ is

$$\frac{1}{12(4\pi f)^2}\sum_{\Xi}\Bigg\{\frac{1}{4}\sum_{\mathcal{S}}\Big[-\ell\left(m^2_{x'\mathcal{S},\Xi}\right)\ -2\ell\left(m^2_{x\mathcal{S},\Xi}\right)-2\ell\left(m^2_{y\mathcal{S},\Xi}\right)\Big]\ (\mathbf{h})+(\mathbf{h}')+(\mathbf{h}'')$$
$$\begin{array}{llll} -\ell(D^{\Xi}_{X'X'}) & (\mathbf{j}) & -2\ell(D^{\Xi}_{XX}) & (\mathbf{n}) \\ -2\ell(D^{\Xi}_{YY}) & (\mathbf{o}) & +c_{\Xi}\ \ell(D^{\Xi}_{XX'}) & (\mathbf{l}) \\ +c_{\Xi}\ \ell(D^{\Xi}_{X'Y}) & (\mathbf{k}) & +3\ell(D^{\Xi}_{XY}) & (\mathbf{m}) \end{array}\Bigg\}\,. \tag{5.11}$$

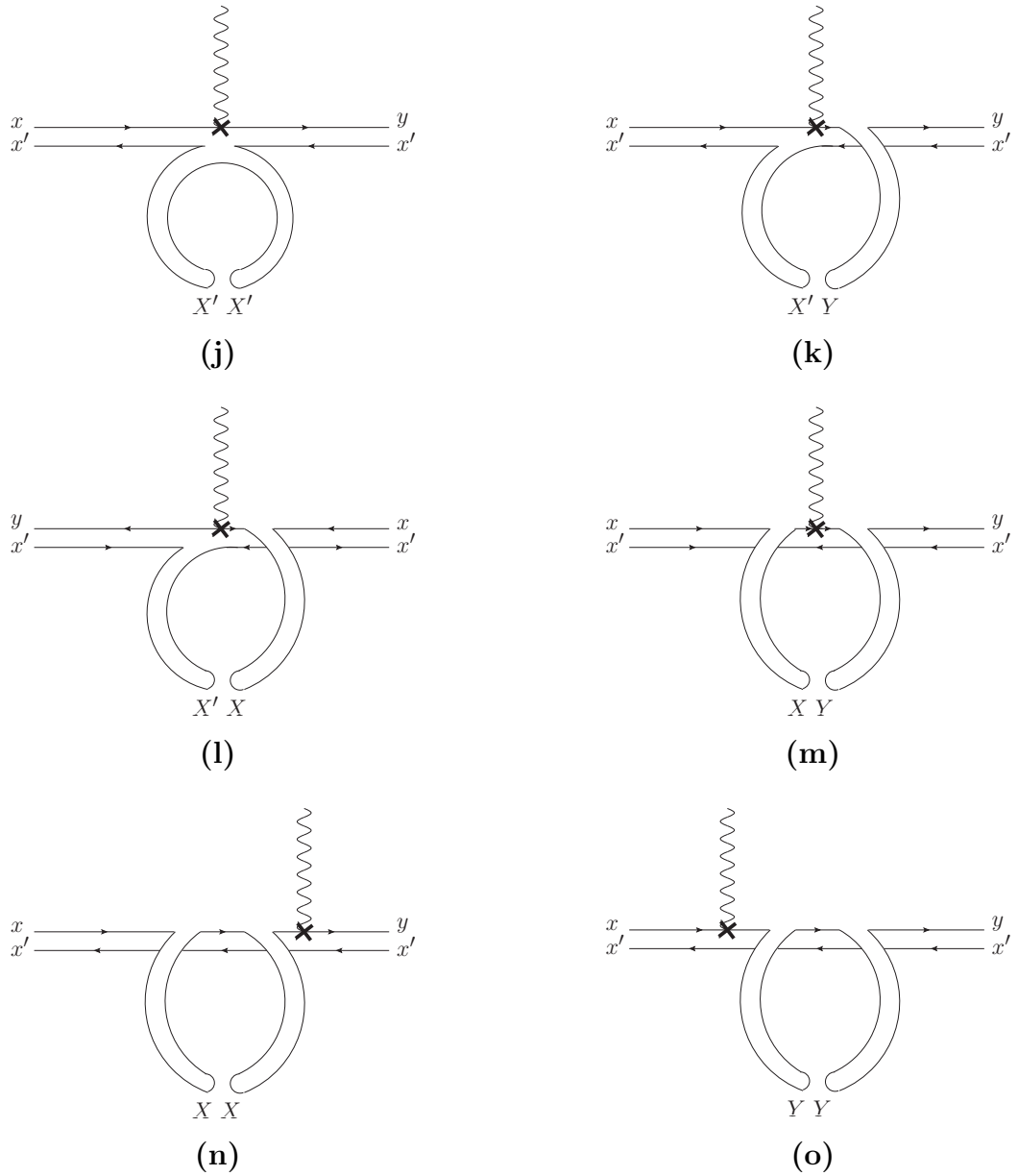


FIG. 5: Quark flow for current vertex diagrams that have disconnected internal meson propagators.

In the $N_f = 1+1+1$ case, Eq. (5.11) becomes, explicitly,

$$
\begin{aligned}
\frac{1}{3(4\pi f)^2}\Bigg\{ & \frac{1}{16}\sum_{\mathcal{S},\Xi}\left[-\ell\left(m^2_{x'\mathcal{S},\Xi}\right)-2\ell\left(m^2_{x\mathcal{S},\Xi}\right)-2\ell\left(m^2_{y\mathcal{S},\Xi}\right)\right] \quad (\mathbf{h})+(\mathbf{h}')+(\mathbf{h}'') \\
+\frac{1}{3}\Bigg[& -\sum_{j\in\mathcal{M}^{(3,x')}}\frac{\partial}{\partial m^2_{X',I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x')};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \quad (\mathbf{j}) \\
& -2\sum_{j\in\mathcal{M}^{(3,x)}}\frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \quad (\mathbf{n}) \\
& -2\sum_{j\in\mathcal{M}^{(3,y)}}\frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \quad (\mathbf{o}) \\
& -\sum_{j\in\mathcal{M}^{(4,x,x')}}R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,x')};\mu_I^{(3)}\right)\ell(m^2_{j,I}) \quad (\mathbf{l}) \\
& -\sum_{j\in\mathcal{M}^{(4,x'y)}}R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x'y)};\mu_I^{(3)}\right)\ell(m^2_{j,I}) \quad (\mathbf{k}) \\
& -3\sum_{j\in\mathcal{M}^{(4,x,y)}}R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I}) \quad (\mathbf{m})\Bigg] \\
+a^2\delta_V\Bigg[& -\sum_{j\in\mathcal{M}^{(4,x')}}\frac{\partial}{\partial m^2_{X',V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x')};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \quad (\mathbf{j}) \\
& -2\sum_{j\in\mathcal{M}^{(4,x)}}\frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \quad (\mathbf{n}) \\
& -2\sum_{j\in\mathcal{M}^{(4,y)}}\frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \quad (\mathbf{o}) \\
& +\sum_{j\in\mathcal{M}^{(5,x,x')}}R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,x')};\mu_V^{(3)}\right)\ell(m^2_{j,V}) \quad (\mathbf{l}) \\
& +\sum_{j\in\mathcal{M}^{(5,x'y)}}R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x'y)};\mu_V^{(3)}\right)\ell(m^2_{j,V}) \quad (\mathbf{k}) \\
& -3\sum_{j\in\mathcal{M}^{(5,x,y)}}R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V}) \quad (\mathbf{m})\Bigg]+\Big[V\to A\Big]\Bigg\}\,. \qquad (5.12)
\end{aligned}
$$

D. Results for $f_2(q^2)$

Adding together Eqs. (5.9) and (5.11) and $(Z_{P_{xx'}} + Z_{P_{yx'}})/2$ from Eq. (5.5), the complete one-loop result for the vector form factor in compact notation is

$$\begin{aligned} f_2(q^2) = -\frac{1}{2(4\pi f)^2} \sum_{\Xi} \Bigg\{ & \frac{1}{16} \sum_{\mathcal{S}} \left[\ell\left(m^2_{x\mathcal{S},\Xi}\right) + \ell\left(m^2_{y\mathcal{S},\Xi}\right) + 4\tilde{B}_{22}(m^2_{x\mathcal{S},\Xi}, m^2_{y\mathcal{S},\Xi}, q^2) \right] \\ & + \frac{1}{4}\left[\ell(D^{\Xi}_{XX}) + \ell(D^{\Xi}_{YY}) - 2\ell(D^{\Xi}_{XY})\right] \\ & + \tilde{B}_{22}(m^2_{xy,\Xi}, D^{\Xi}_{XX}, q^2) + \tilde{B}_{22}(m^2_{xy,\Xi}, D^{\Xi}_{YY}, q^2) \\ & - 2\tilde{B}_{22}(m^2_{xy,\Xi}, D^{\Xi}_{XY}, q^2) \Bigg\} , \end{aligned} \tag{5.13}$$

where Ξ runs over the sixteen independent meson tastes and $\mathcal{S}$ runs over the three sea quark flavors. Note that the answer is independent of the spectator quark mass $m_{x'}$ at this order. The $m_{x'}$ dependence of the current vertex contribution cancels corresponding contributions from the wave function renormalization terms. This appears to be necessary in order to satisfy the AG theorem, which is a statement about the dependence on the valence quark masses m_x and m_y. Indeed, it is not hard to check that Eq. (5.13) obeys the AG theorem: As functions of the disconnected propagators, ℓ and $\tilde{B}_{22}$ are linear. Combining $D^{\Xi}_{XX}+D^{\Xi}_{YY}-2D^{\Xi}_{XY}$ using Eq. (3.10), one easily extracts an overall factor of $(m^2_{Y,\Xi}-m^2_{X,\Xi})^2 \propto (m_y - m_x)^2$ from the disconnected terms (the taste splittings cancel in this difference). For the connected terms, those that are summed over $\mathcal{S}$, we may use Eq. (A12) to show that the contribution for each $\mathcal{S}$ is proportional to $(m^2_{y\mathcal{S}} - m^2_{x\mathcal{S}})^2 \propto (m_y - m_x)^2$ as $m_y \to m_x$. Thus $f_2(0)$ is second order in $m_y - m_x$ as required by the AG theorem.

We note also that Eq. (5.13) is independent of the taste of the external mesons; the taste-dependent factors c_Ξ in Eqs. (5.5) and (5.11) have canceled. Recall that we have taken the weak current to be a taste singlet (see Eq. (5.2)). With a nonsinglet taste structure for the current, the result would of course depend on the taste of the mesons.

With the sea quark masses m_u, m_d, and m_s nondegenerate (the 1+1+1 case), the result

Eq. (5.13) becomes, explicitly,

$$
\begin{aligned}
f_2^{N_f=1+1+1}(q^2) \;=\; & -\frac{1}{2(4\pi f)^2}\Bigg\{\frac{1}{16}\sum_{\mathcal{S},\Xi}\left[\ell\left(m^2_{x\mathcal{S},\Xi}\right)+\ell\left(m^2_{y\mathcal{S},\Xi}\right)+4\,\tilde{B}_{22}(m^2_{x\mathcal{S},\Xi},m^2_{y\mathcal{S},\Xi},q^2)\right] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right)+2\sum_j R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I}) \\
& +4\frac{\partial}{\partial m^2_{X,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\right) \\
& +4\frac{\partial}{\partial m^2_{Y,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\Bigg] \\
& +a^2\delta_V\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right)+2\sum_j R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V}) \\
& +4\frac{\partial}{\partial m^2_{X,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\right) \\
& +4\frac{\partial}{\partial m^2_{Y,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\Bigg]+\Big[V\to A\Big]\Bigg\}\,. \qquad (5.14)
\end{aligned}
$$

Keeping the valence masses arbitrary, but assuming exact isospin in the sea ($m_u = m_d$), we can also find the explicit $N_f = 2+1$ expression given in Appendix B. The partially quenched continuum results for the $N_f = 1+1+1$ and $N_f = 2+1$ cases are also given in Appendix B.

VI. RESULTS FOR $f_+(q^2)$ IN THE MIXED-ACTION CASE

The only explicit difference between the mixed-action case and the unmixed theory discussed in the previous section is that mixed-action disconnected propagator for vector- and axial-tastes has the form given in Eq. (4.17), rather than Eq. (3.10). Of course there are also implicit differences in the values the meson masses for various tastes: taste splittings for the valence (HISQ), sea (asqtad), and mixed mesons are given by $\Delta_\Xi^{\rm HISQ}$, $\Delta_\Xi^{\rm asqtad}$ and $\Delta_\Xi^{\rm mix}$,

respectively. With these caveats, one may use the result given in Eq. (5.13) for the mixed case, just as for the unmixed case. More explicitly, for $N_f = 1+1+1$ in the mixed-action case we have

$$
\begin{aligned}
f_2^{N_f=1+1+1}(q^2) \;=\; & -\frac{1}{2(4\pi f)^2}\Bigg\{\frac{1}{16}\sum_{\mathcal{S},\Xi}\Big[\ell\left(m^2_{x\mathcal{S},\Xi}\right)+\ell\left(m^2_{y\mathcal{S},\Xi}\right)+4\,\tilde{B}_{22}(m_{x\mathcal{S},\Xi},m_{y\mathcal{S},\Xi},q^2)\Big] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I})\right)+2\sum_j R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\ell(m^2_{j,I}) \\
& +4\frac{\partial}{\partial m^2_{X,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,3]}\left(\mathcal{M}_I^{(3,x)};\mu_I^{(3)}\right)\right) \\
& +4\frac{\partial}{\partial m^2_{Y,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,3]}\left(\mathcal{M}_I^{(3,y)};\mu_I^{(3)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[4,3]}\left(\mathcal{M}_I^{(4,x,y)};\mu_I^{(3)}\right)\Bigg] \\
& +a^2(\delta_V^{vv}-\delta_V^{\rm mix})\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V})\right)+2\sum_j R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\ell(m^2_{j,V}) \\
& +4\frac{\partial}{\partial m^2_{X,V}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,3]}\left(\mathcal{M}_V^{(4,x)};\mu_V^{(3)}\right)\right\} \\
& +4\frac{\partial}{\partial m^2_{Y,V}}\left\{\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,3]}\left(\mathcal{M}_V^{(4,y)};\mu_V^{(3)}\right)\right\} \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[5,3]}\left(\mathcal{M}_V^{(5,x,y)};\mu_V^{(3)}\right)\Bigg] \\
& +a^2\delta_V^{mix}\Bigg[\frac{\partial\ell(m^2_{X,V})}{\partial m^2_{X,V}}+\frac{\partial\ell(m^2_{Y,V})}{\partial m^2_{Y,V}}+2\,\frac{\left(\ell(m^2_{X,V})-\ell(m^2_{Y,V})\right)}{m^2_{Y,V}-m^2_{X,V}} \\
& +4\,\frac{\partial}{\partial m^2_{X,V}}\tilde{B}_{22}(m^2_{xy,V},m^2_{X,V},q^2)+4\,\frac{\partial}{\partial m^2_{Y,V}}\tilde{B}_{22}(m^2_{xy,V},m^2_{Y,V},q^2) \\
& +\frac{8}{m^2_{Y,V}-m^2_{X,V}}\left(\tilde{B}_{22}(m^2_{xy,V},m^2_{X,V},q^2)-\tilde{B}_{22}(m^2_{xy,V},m^2_{Y,V},q^2)\right)\Bigg] \\
& +\Big[V\to A\Big]\Bigg\}\,, \qquad (6.1)
\end{aligned}
$$

where again Ξ runs over the sixteen independent meson tastes and $\mathcal{S}$ runs over sea quark flavors. Since the Ademollo-Gatto theorem is obeyed at the level of Eq. (5.13), this mixed-action version must also obey the theorem. It is also easy to see directly that the terms proportional to $\delta^{mix}_{A,V}$, which contain the mixed-action effects, vanish as $(m_y - m_x)^2$ as $m_y \to m_x$.

If we take the isospin limit ($m_u = m_d$) of the expression above, the $N_f = 2+1$ case, we obtain the result quoted in Eq. (B4) of Appendix B. That result has already been given in Ref. [16] for the specific case $q^2 = 0$.

VII. CONCLUSIONS

We have calculated $f_2(q^2)$, the one-loop, non-analytic part of the vector form factor, in partially quenched SχPT for $N_f = 1{+}1{+}1$, as well as in the isospin limit, $N_f = 2{+}1$. The complete vector form factor to NLO is given by $f_+(q^2) = 1 + f_2(q^2) + \frac{4}{f^2} L^r_9(\mu) q^2$, where the last term is the analytic contribution, which vanishes at $q^2 = 0$. We incorporate staggered effects for both the case where the action in the sea and the valence sectors are different (mixed-action) and the case in which all quarks are described with the same staggered action. We have found that, at this order, the form factor is independent of the spectator mass quark for any value of the momentum transfer. We also confirm that all our expressions obey the Ademollo-Gatto theorem.

The $N_f = 2{+}1$ mixed-action expression in (B4) was used in the determination of $|V_{us}|$ by the Fermilab Lattice/MILC Collaboration in Ref. [16], and the $N_f = 2{+}1$ unmixed expresion in (B1) is used in the recent analysis by the same collaboration in Ref. [19]. In this second work, the Fermilab Lattice/MILC Collaboration provides the most precise determination of $f_+(0)$, and the first one including simulations at the physical light quark masses.

Even with simulations at the physical light quark masses, χPT is still very useful in miminizing the errors. The χPT formulation allows one to incorporate (more precise) data at heavier masses and correct for mistunings of the quark masses in the simulations, both in the valence and in the sea sector. It is also the perfect framework to incorporate analytically and systematically the corrections associated to the lattice artifacts, such as discretization and finite volume effects. This can be done for a specific lattice fermion formulation, as we have done here for staggered actions. Finite volume effects are now one of the dominant sources of error [19], so it is crucial to include those corrections in our PQSχPT formulae [49] in order to achieve the 0.2% precision in $f_+(0)$ required by the size of the experimental uncertainties.

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Appendix A: One-loop integrals

We need the following two Euclidean integrals

$$\mathcal{A}(m^2) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} , \tag{A1}$$

$$\mathcal{B}_{\mu\nu}(m_1^2, m_2^2, q^2) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2+m_1^2)((k-q)^2+m_2^2)} , \tag{A2}$$

$$= q_\mu q_\nu \mathcal{C}_{21}(m_1^2, m_2^2, q^2) - \delta_{\mu\nu}\mathcal{C}_{22}(m_1^2, m_2^2, q^2) \tag{A3}$$

After regularization and renormalization following Ref. [48], one has

$$\mathcal{A}(m^2) \to \frac{1}{16\pi^2}\, \ell(m^2) , \tag{A4}$$

$$\mathcal{C}_{22}(m_1^2, m_2^2, q^2) \to \frac{1}{16\pi^2}\, \tilde{B}_{22}(m_1^2, m_2^2, -q^2) \tag{A5}$$

The chiral logarithm function ℓ is given by

$$\ell(m^2) \equiv m^2 \ln(m^2/\Lambda_\chi^2) , \tag{A6}$$

with Λ_χ the chiral scale. The function $\tilde{B}_{22}(m_1^2, m_2^2, s)$ is related to the function $\bar{B}_{22}$ defined in Ref. [48] by

$$\tilde{B}_{22}(m_1^2, m_2^2, s) = (4\pi)^2\, \bar{B}_{22}(m_1^2, m_2^2, s, \Lambda_\chi^2) . \tag{A7}$$

The minus sign for the q^2 argument in Eq. (A5) arises because q in Eq. (A2) is defined to be a Euclidean momentum; the physical momentum transfer squared is $s = -q^2$. Explicitly, $\tilde{B}_{22}$ is given by

$$\tilde{B}_{22}(m_1^2, m_2^2, s) = \frac{1}{6}\left[-\ell(m_2^2) + 2m_1^2\tilde{B}(m_1^2, m_2^2, s) - (s+m_1^2-m_2^2)\tilde{B}_1(m_1^2, m_2^2, s)\right] + \frac{1}{18}\left[3m_1^2 + 3m_2^2 - s\right] , \tag{A8}$$

where

$$\tilde{B}(m_1^2, m_2^2, s) = \frac{1}{2}\left[2 + \left(\frac{\Sigma}{\Delta} - \frac{\Delta}{s}\right)\ln\frac{m_1^2}{m_2^2} - \frac{\nu}{s}\ln\frac{(s+\nu)^2-\Delta^2}{(s-\nu)^2-\Delta^2}\right] - \frac{\ell(m_1^2)-\ell(m_2^2)}{m_1^2-m_2^2} , \tag{A9}$$

$$\tilde{B}_1(m_1^2, m_2^2, s) = \frac{1}{2s}\Big(\ell(m_1^2) - \ell(m_2^2) + (m_1^2 - m_2^2 + s)\tilde{B}(m_1^2, m_2^2, s)\Big) , \tag{A10}$$

with $\Delta = m_1^2 - m_2^2$, $\Sigma = m_1^2 + m_2^2$, and $\nu^2 = [s-(m_1+m_2)^2][s-(m_1-m_2)^2]$.

In the special case of $s = q^2 = 0$, $\tilde{B}_{22}$ takes the simple form

$$\tilde{B}_{22}(m_1^2, m_2^2, 0) = -\frac{1}{4}\left(\frac{m_2^2\,\ell(m_2^2) - m_1^2\,\ell(m_1^2)}{m_2^2 - m_1^2}\right) + \frac{1}{8}\left(m_1^2 + m_2^2\right) \ . \tag{A11}$$

For checking the AG theorem, we need the behavior of $\tilde{B}_{22}(m_1^2, m_2^2, 0)$ as $m_2^2 \to m_1^2$. Letting $m_1^2 = m^2$, $m_2^2 = m^2 + \epsilon$, and expanding Eq. (A11) through $\mathcal{O}(\epsilon)$, we find

$$\tilde{B}_{22}(m^2, m^2 + \epsilon, 0) = -\frac{1}{4}\Big[\ln(m^2/\Lambda_\chi^2)(2m^2 + \epsilon) + \epsilon\Big] + \mathcal{O}(\epsilon^2) \ . \tag{A12}$$

For diagrams with neutral particles in the loop, the simple propagator in Eq. (A1), or one of the two propagators in Eq. (A2), may be replaced by a disconnected propagator, Eq. (3.10) or Eq. (3.11). For an explicit representation of the result of the integrals, one may follow the standard procedure and write the disconnected propagator as a sum over residues times simple poles, and apply Eqs. (A4) and (A5). However, this produces complicated expressions that depend on the details of the sea sector (*e.g.*, are different in the 1+1+1 case and the 2+1 case, as well in the mixed-action and unmixed cases). To see the overall structure of the results more clearly and compactly, it is useful also to have a notation that writes the integrals as functions of the disconnected propagator itself. Thus we allow a replacement of the argument m^2 in Eq. (A1) or m_2^2 in Eq. (A2) by a disconnected propagator and define

$$\mathcal{A}(D^\Xi) \ \equiv \ \int \frac{d^4k}{(2\pi)^4}\, D^\Xi(k) \ , \tag{A13}$$

$$\mathcal{B}_{\mu\nu}(m_1^2, D^\Xi, q^2) \ \equiv \ \int \frac{d^4k}{(2\pi)^4}\, \frac{k_\mu k_\nu}{(k^2 + m_1^2)} D^\Xi(k - q) \ , \tag{A14}$$

$$= \ q_\mu q_\nu \mathcal{C}_{21}(m_1^2, D^\Xi, q^2) - \delta_{\mu\nu}\mathcal{C}_{22}(m_1^2, D^\Xi, q^2) \tag{A15}$$

The corresponding expressions after regularization and renormalization are then denoted as

$$\mathcal{A}(D^\Xi) \ \to \ \frac{1}{16\pi^2}\, \ell(D^\Xi) \ , \tag{A16}$$

$$\mathcal{C}_{22}(m_1^2, D^\Xi, q^2) \ \to \ \frac{1}{16\pi^2}\, \tilde{B}_{22}(m_1^2, D^\Xi, -q^2) \ . \tag{A17}$$

Appendix B: Form factor in the isospin limit $N_f = 2 + 1$ and in the continuum

In this Appendix we collect the isospin limit and the continuum limit of the one-loop PQSχPT in Eq. (5.14) and Eq. (6.1).

1. Unmixed case

The vector form factor at one loop in PQSχPT and in the isospin limit, $N_f = 2+1$ is

$$
\begin{aligned}
f_2^{N_f=2+1}(q^2) \;=\; & -\frac{1}{2(4\pi f)^2}\Bigg\{\frac{1}{16}\sum_{\mathcal{S},\Xi}\left[\ell\left(m^2_{x\mathcal{S},\Xi}\right)+\ell\left(m^2_{y\mathcal{S},\Xi}\right)+4\,\tilde{B}_{22}(m_{x\mathcal{S},\Xi},m_{y\mathcal{S},\Xi},q^2)\right] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[2,2]}\left(\mathcal{M}_I^{(2,x)};\mu_I^{(2)}\right)\ell(m^2_{j,I})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[2,2]}\left(\mathcal{M}_I^{(2,y)};\mu_I^{(2)}\right)\ell(m^2_{j,I})\right)+2\sum_j R_j^{[3,2]}\left(\mathcal{M}_I^{(3,x,y)};\mu_I^{(2)}\right)\ell(m^2_{j,I}) \\
& +4\frac{\partial}{\partial m^2_{X,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[2,2]}\left(\mathcal{M}_I^{(2,x)};\mu_I^{(2)}\right)\right) \\
& +4\frac{\partial}{\partial m^2_{Y,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[2,2]}\left(\mathcal{M}_I^{(2,y)};\mu_I^{(2)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,2]}\left(\mathcal{M}_I^{(3,x,y)};\mu_I^{(2)}\right)\Bigg] \\
& +a^2\delta_V\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[3,2]}\left(\mathcal{M}_V^{(3,x)};\mu_V^{(2)}\right)\ell(m^2_{j,V})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[3,2]}\left(\mathcal{M}_V^{(3,y)};\mu_V^{(2)}\right)\ell(m^2_{j,V})\right)+2\sum_j R_j^{[4,2]}\left(\mathcal{M}_V^{(4,x,y)};\mu_V^{(2)}\right)\ell(m^2_{j,V}) \\
& +4\frac{\partial}{\partial m^2_{X,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[3,2]}\left(\mathcal{M}_V^{(3,x)};\mu_V^{(2)}\right)\right) \\
& +4\frac{\partial}{\partial m^2_{Y,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[3,2]}\left(\mathcal{M}_V^{(3,y)};\mu_V^{(2)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,2]}\left(\mathcal{M}_V^{(4,x,y)};\mu_V^{(2)}\right)\Bigg]+\Big[V\to A\Big]\Bigg\}\,,
\end{aligned}
\tag{B1}
$$

where the mass sets $\mathcal{M}^{(2,x)}$, $\mathcal{M}^{(3,x,y)}$, ... correspond to those of Eq. (5.8) but with the π^0 mass eliminated and the first subscript reduced by 1. Similarly, the set $\mu^{(2)}$ corresponds to $\mu^{(3)}$ in Eq. (5.8), with the D mass eliminated.

In the continuum, the partially quenched $N_f = 1+1+1$ result in Eq. (5.14) reduces to

$$
\begin{aligned}
f_2^{N_f=1+1+1}(q^2) \;=\; & -\frac{1}{2(4\pi f)^2}\Bigg\{\sum_{\mathcal{S}}\left[\ell\left(m_{x\mathcal{S}}^2\right)+\ell\left(m_{y\mathcal{S}}^2\right)+4\tilde{B}_{22}(m_{x\mathcal{S}},m_{y\mathcal{S}},q^2)\right] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m_X^2}\left(R_j^{[3,3]}\left(\mathcal{M}^{(3,x)};\mu^{(3)}\right)\ell(m_j^2)\right) \\
& +\sum_j \frac{\partial}{\partial m_Y^2}\left(R_j^{[3,3]}\left(\mathcal{M}^{(3,y)};\mu^{(3)}\right)\ell(m_j^2)\right)+2\sum_j R_j^{[4,3]}\left(\mathcal{M}^{(4,x,y)};\mu^{(3)}\right)\ell(m_j^2) \\
& +4\frac{\partial}{\partial m_X^2}\left(\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[3,3]}\left(\mathcal{M}^{(3,x)};\mu^{(3)}\right)\right) \\
& +4\frac{\partial}{\partial m_Y^2}\left(\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[3,3]}\left(\mathcal{M}^{(3,y)};\mu^{(3)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[4,3]}\left(\mathcal{M}^{(4,x,y)};\mu^{(3)}\right)\Bigg]\Bigg\}\,,
\end{aligned}
\tag{B2}
$$

where the meson masses m_{xy}^2 are those in the continuum, the residue functions $R^{[n,k]}(\mathcal{M};\mu)$ are the same as in the staggered expressions, and $\mathcal{M}$ and μ are the set of meson masses defined in Eq. (5.8) but in the continuum.

In the partially quenched $N_f = 2+1$ case, the result in Eq. (B1) reduces in the continuum to

$$
\begin{aligned}
f_2^{N_f=2+1}(q^2) \;=\; & -\frac{1}{2(4\pi f)^2}\Bigg\{\sum_{\mathcal{S}}\left[\ell\left(m_{x\mathcal{S}}^2\right)+\ell\left(m_{y\mathcal{S}}^2\right)+4\tilde{B}_{22}(m_{x\mathcal{S}},m_{y\mathcal{S}},q^2)\right] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m_X^2}\left(R_j^{[2,2]}\left(\mathcal{M}^{(2,x)};\mu^{(2)}\right)\ell(m_j^2)\right) \\
& +\sum_j \frac{\partial}{\partial m_Y^2}\left(R_j^{[2,2]}\left(\mathcal{M}^{(2,y)};\mu^{(2)}\right)\ell(m_j^2)\right)+2\sum_j R_j^{[3,2]}\left(\mathcal{M}^{(3,x,y)};\mu^{(2)}\right)\ell(m_j^2) \\
& +4\frac{\partial}{\partial m_X^2}\left(\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[2,2]}\left(\mathcal{M}^{(2,x)};\mu^{(2)}\right)\right) \\
& +4\frac{\partial}{\partial m_Y^2}\left(\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[2,2]}\left(\mathcal{M}^{(2,y)};\mu^{(2)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m_{xy}^2,m_j^2,q^2)R_j^{[3,2]}\left(\mathcal{M}^{(3,x,y)};\mu^{(2)}\right)\Bigg]\Bigg\}\,.
\end{aligned}
\tag{B3}
$$

The continuum χPT result $f_2^{N_f=2+1}$ for zero momentum transfer, $q^2 = 0$, was already given in the Appendix of Ref. [36], but we believe there was a misprint affecting the sign of the last term in second line of preprint version.

2. Mixed-action case

Taking the limit $m_u = m_d$ (the 2+1 case) of the mixed action result, Eq. (6.1), we get

$$
\begin{aligned}
f_2^{N_f=2+1}(q^2) \;=\; & \frac{-1}{2(4\pi f)^2}\Bigg\{ \frac{1}{16}\sum_{\mathcal{S},\Xi}\left[\ell\left(m^2_{x\mathcal{S},\Xi}\right)+\ell\left(m^2_{y\mathcal{S},\Xi}\right)+4\tilde{B}_{22}(m_{x\mathcal{S}},m_{y\mathcal{S}},q^2)\right] \\
& +\frac{1}{3}\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,I}}\left(R_j^{[2,2]}\left(\mathcal{M}_I^{(2,x)};\mu_I^{(2)}\right)\ell(m^2_{j,I})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,I}}\left(R_j^{[2,2]}\left(\mathcal{M}_I^{(2,y)};\mu_I^{(2)}\right)\ell(m^2_{j,I})\right)+2\sum_j R_j^{[3,2]}\left(\mathcal{M}_I^{(3,x,y)};\mu_I^{(2)}\right)\ell(m^2_{j,I})\Bigg] \\
& +\frac{4}{3}\,\frac{\partial}{\partial m^2_{X,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[2,2]}\left(\mathcal{M}_I^{(2,x)};\mu_I^{(2)}\right)\right) \\
& +\frac{4}{3}\,\frac{\partial}{\partial m^2_{Y,I}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[2,2]}\left(\mathcal{M}_I^{(2,y)};\mu_I^{(2)}\right)\right) \\
& +\frac{8}{3}\sum_j \tilde{B}_{22}(m^2_{xy,I},m^2_{j,I},q^2)R_j^{[3,2]}\left(\mathcal{M}_I^{(3,x,y)};\mu_I^{(2)}\right) \\
& +a^2(\delta_V^{vv}-\delta_V^{mix})\Bigg[\sum_j \frac{\partial}{\partial m^2_{X,V}}\left(R_j^{[3,2]}\left(\mathcal{M}_V^{(3,x)};\mu_V^{(2)}\right)\ell(m^2_{j,V})\right) \\
& +\sum_j \frac{\partial}{\partial m^2_{Y,V}}\left(R_j^{[3,2]}\left(\mathcal{M}_V^{(3,y)};\mu_V^{(2)}\right)\ell(m^2_{j,V})\right)+2\sum_j R_j^{[4,2]}\left(\mathcal{M}_V^{(4,x,y)};\mu_V^{(2)}\right)\ell(m^2_{j,V}) \\
& +4\,\frac{\partial}{\partial m^2_{X,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[3,2]}\left(\mathcal{M}_V^{(3,x)};\mu_V^{(2)}\right)\right) \qquad \text{(B4)} \\
& +4\,\frac{\partial}{\partial m^2_{Y,V}}\left(\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[3,2]}\left(\mathcal{M}_V^{(3,y)};\mu_V^{(2)}\right)\right) \\
& +8\sum_j \tilde{B}_{22}(m^2_{xy,V},m^2_{j,V},q^2)R_j^{[4,2]}\left(\mathcal{M}_V^{(4,x,y)};\mu_V^{(2)}\right)\Bigg] \\
& +a^2\delta_V^{mix}\Bigg[\frac{\partial\ell(m^2_{X,V})}{\partial m^2_{X,V}}+\frac{\partial\ell(m^2_{Y,V})}{\partial m^2_{Y,V}}+2\,\frac{\left(\ell(m^2_{X,V})-\ell(m^2_{Y,V})\right)}{m^2_{Y,V}-m^2_{X,V}} \\
& +4\,\frac{\partial}{\partial m^2_{X,V}}\tilde{B}_{22}(m^2_{xy,V},m^2_{X,V},q^2)+4\,\frac{\partial}{\partial m^2_{Y,V}}\tilde{B}_{22}(m^2_{xy,V},m^2_{Y,V},q^2) \\
& +\frac{8}{m^2_{Y,V}-m^2_{X,V}}\left(\tilde{B}_{22}(m^2_{xy,V},m^2_{X,V},q^2)-\tilde{B}_{22}(m^2_{xy,V},m^2_{Y,V},q^2)\right)\Bigg] \\
& +\Big[V\to A\Big]\Bigg\}\,,
\end{aligned}
$$

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