PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Center of the Dihedral:** The *center* $C(G)$ of a group G (discrete or continuous) is the set of elements that commute with every element of G:

$$C(G) \;=\; \{a \in G | ag = ga \;\forall\; g \in G\} \;. \tag{1}$$

Now consider the Dihedral group D_4 (symmetry group of the square) as an example:

   (a) What is the center C of D_4?

   (b) What is the quotient group D_4/C?

   (c) Is D_4 isomorphic to $(D_4/C) \otimes C$?

2. **Equivalence Class of $SO(3)$:** Show that $SO(3)$ rotations around different axes but with the same angle are *equivalent*, i.e.[1]

$$e^{-i\phi\hat{n}_2\cdot\vec{J}}\, e^{i\theta\hat{n}_1\cdot\vec{J}}\, e^{i\phi\hat{n}_2\cdot\vec{J}} \;=\; e^{i\theta\hat{n}_3\cdot\vec{J}}, \tag{2}$$

where $\hat{n}_3 = R(\hat{n}_2, \phi)\hat{n}_1$ and $R(\hat{n}_2, \phi)$ represents the $SO(3)$ rotation around the unit vector $\hat{n}_2$ through an angle ϕ.

3. **Electric Quadrupole:** The components of the electric quadrupole operator $\boldsymbol{D}$ are defined as

$$D_{ij} \;=\; 3r_ir_j - r^2\delta_{ij}\,, \tag{3}$$

where r_i (with $i = 1, 2, 3$) are the spatial (x, y, z) coordinates of the electron.

   (a) Determine the commutation relations $[J_i, D_{jk}]$, where J_i's are the angular momentum operators.

   (b) Show that there are only five independent components of $\boldsymbol{D}$, which transform under $SO(3)$ rotation as a $j = 2$ representation.

[1] Recall the definition that in a group G, two elements g and g' are said to be equivalent if there exists another element f such that $g' = f^{-1}gf$.

(c) For a single electron transition from one atomic orbital labeled by $|n, j, m\rangle$ to another labeled by $|n', j', m'\rangle$ (which are eigenstates of J_3 with eigenvalues m and m' respectively) due to the application of $\boldsymbol{D}$, what are the selection rules for $\Delta j = j' - j$ and $\Delta m = m' - m$? (*Hint:* Use Wigner-Eckart theorem.)

4. $SU(2)$ **Matrices:** We argued that any 2×2 special unitary matrix can be written as a linear combination of the Pauli matrices. Let us demonstrate it in another way:

(a) Start with an arbitrary complex 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{4}$$

Now impose the special unitarity condition that $M^\dagger M = \mathbb{1}$ and $\det M = 1$. Show that it reduces the number of real parameters from 8 to just 3.

(b) Call them (r, ϕ, φ). Show that M can be written as

$$M = \begin{pmatrix} \sqrt{1-r^2}e^{i\phi} & re^{i\varphi} \\ -re^{-i\varphi} & \sqrt{1-r^2}e^{-i\phi} \end{pmatrix} . \tag{5}$$

(c) Show that Eq. (5) is the same as the more familiar $SU(2)$ element

$$U = e^{i\theta\hat{n}\cdot\vec{\sigma}/2} , \tag{6}$$

where σ_i (with $i = 1, 2, 3$) are the three Pauli matrices. Identify θ and $\hat{n}$ in terms of r, ϕ, φ.