PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **3-D harmonic oscillator:** The 3-dimensional harmonic oscillator is described by the Hamiltonian

$$H \;=\; \frac{\mathbf{p}^2}{2m} + \frac{1}{2}k\mathbf{r}^2 \;=\; \sum_{i=1}^{3}\left(a_i^\dagger a_i + \frac{1}{2}\right)\hbar\omega\,. \tag{1}$$

   (a) What are the energy eigenvalues?

   (b) You will probably notice that the spectrum has a higher degree of degeneracy than that predicted by mere rotational symmetry $SO(3)$. Show that this higher symmetry is $U(3) = SU(3) \otimes U(1)$. What does the $U(1)$ describe here?

   (c) The eigenstates with energy E_n are given by $a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle$, which manifestly transform like an $SU(3)$ tensor with n lower (or upper) indices. Calculate the dimension d of this tensor.

   (d) Show that d from part (c) gives the degree of degeneracy of the energy eigenvalues you obtained in part (a). This gives another example of the power of symmetry.

2. **Generalized Isospin:** In class, we discussed the three overlapping $SU(2)$ subalgebras of $SU(3)$. These are referred to as the generalized isospin, or I-spin, U-spin, and V-spin, generated by $(I_3, I_\pm)$, $(U_3, U_\pm)$, and $(V_3, V_\pm)$, respectively, where

$$I_\pm = T_1 \pm iT_2\,, \quad U_\pm = T_6 \pm iT_7\,, \quad \text{and} \quad V_\pm = T_4 \pm iT_5\,, \tag{2}$$

   T_a's being the $SU(3)$ generators. Using the $SU(3)$ Lie algebra

$$[T_a, T_b] \;=\; if_{abc}T_c \tag{3}$$

   and the structure constants you derived in HW7, problem 3a, work out *all* the commutation relations among these generalized isospin generators. These results will be used in our discussion of roots and weights – a central concept in Lie algebra.

3. **Weight Diagram:** Following the discussion in class, draw the weight diagram for the decuplet **10** of $SU(3)$. Remember to write the coordinates of each lattice point.