PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Clebsch-Gordan decomposition:** Using the Clebsch-Gordan (CG) decomposition of $j \otimes j'$ in the form

$$|J, M\rangle \;=\; \sum_{m=-j}^{j} \sum_{m'=-j'}^{j'} |j, j', m, m'\rangle\langle j, j', m, m'|J, M\rangle \,, \tag{1}$$

work out the CG coefficients for $j = 1$ and $j' = \frac{1}{2}$ and show that

$$1 \otimes \frac{1}{2} \;=\; \frac{3}{2} \oplus \frac{1}{2} \,. \tag{2}$$

In terms of the dimensions of the corresponding irreps [of $SU(2)$], this means

$$\mathbf{2} \otimes \mathbf{3} \;=\; \mathbf{4} \oplus \mathbf{2} \,. \tag{3}$$

2. **Another useful identity for determinants:**

   (a) Show that for any diagonalizable matrix M,

$$\det M \;=\; e^{\mathrm{Tr}(\log M)} \,. \tag{4}$$

   We have used this identity to show that the generators of $SU(N)$ are traceless Hermitian matrices.

   (b) Now if $M \to M + \delta M$, using Eq. (4) show that

$$\delta(\det M) \;=\; (\det M) \sum_{ij} (M^{-1})_{ij} (\delta M)_{ji} \,. \tag{5}$$

This actually furnishes a formula for the inverse of a matrix.

3. **Pauli and Gell-Mann:**

   (a) Using the Pauli and Gell-Mann matrices, calculate the structure constants for $SU(2)$ and $SU(3)$, respectively.

(b) Show that the Pauli and Gell-Mann matrices respectively satisfy the anti-commutation relations

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}\,, \tag{6}$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c\,, \tag{7}$$

where $j, k = 1, 2, 3$, and $a, b, c, = 1, 2, \cdots, 8$ and the d-symbols in Eq. (7) are a bunch of *real* numbers (evaluate them!). Can you tell the main difference between these d_{abc} and the $SU(3)$ structure constants f_{abc} (apart from the fact that they are not identical)?

(c) Using Eqs. (6) and (7), verify that the Pauli and Gell-Mann matrices satisfy

$$\mathrm{Tr}\left(\frac{\sigma_j}{2}\frac{\sigma_k}{2}\right) = \frac{1}{2}\delta_{jk}\,, \tag{8}$$

$$\mathrm{Tr}\left(\frac{\lambda_a}{2}\frac{\lambda_b}{2}\right) = \frac{1}{2}\delta_{ab}\,. \tag{9}$$

In general, all $SU(N)$ generators T_a are normalized by $\mathrm{Tr}(T_aT_b) = \frac{1}{2}\delta_{ab}$.