PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Spherical harmonics:** This warm-up exercise is to refresh your memory on orbital angular momentum operators in quantum mechanics.[1]

   (a) Applying the Heisenberg prescription $\vec{p} \to -i\hbar\,\vec{\nabla}$ to the classical angular momentum $\vec{L} = \vec{x} \times \vec{p}$, we can write it as a differential operator: $\vec{L} = -i\hbar\,\vec{x} \times \vec{\nabla}$. Write down the explicit forms of L_z, $L_\pm = L_x \pm iL_y$ and $\vec{L}^2$ in spherical polar coordinates (r, θ, ϕ).

   (b) The eigenvalue equations for L_z and $\vec{L}^2$ can be written in terms of the *spherical harmonics* $Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m \rangle$:

$$L_z Y_l^m(\theta, \phi) = m\hbar\, Y_l^m(\theta, \phi)\,, \tag{1}$$

$$\vec{L}^2 Y_l^m(\theta, \phi) = l(l+1)\hbar^2\, Y_l^m(\theta, \phi)\,, \tag{2}$$

   where $l = 0, 1, 2, \cdots$ is a non-negative integer[2] and $m = -l, -l+1, \cdots, l-1, l$. Using Eq. (1) and the results from part (a), show that the appropriately normalized eigenfunctions $Y_l^m(\theta, \phi)$ are given by[3]

$$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta) e^{im\phi} \tag{3}$$

   for $m \geq 0$ and $Y_l^{-m} = (-1)^m (Y_l^m)^*$. Here $P_l^0(\cos\theta) \equiv P_l(\cos\theta)$ are the *Legendre polynomials* and $P_l^m(\cos\theta)$ are the *associated* Legendre polynomials, given by (for $-l \leq m \leq l$)

$$P_l^m(u) = \frac{(1-u^2)^{m/2}}{2^l l!}\left(\frac{d}{du}\right)^{l+m} (u^2-1)^l\,. \tag{4}$$

   You might find it useful (for the next problem) to list a few of these explicitly!

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[1] Those who have never seen this stuff before, please refer to any undergrad quantum mechanics textbook, e.g. D. J. Griffiths, *Introduction to Quantum Mechanics*, Prentice Hall (1995).

[2] *Teaser:* What happens if we include half-odd integers, i.e. $l = 1/2, 3/2, \cdots$?

[3] *Hint:* This involves several steps: (i) Use separation of variables and operate by L_z to extract the ϕ dependence; (ii) use $L_\pm Y_l^{\pm l}(\theta, \phi) = 0$ to extract the θ dependence; (iii) use the normalization condition $\langle Y_l^m | Y_l^m \rangle = 1$ to fix the integration constant.

2. **Why do we care so much about symmetric traceless tensors?**

   (a) Consider the symmetric traceless tensors $S_{i_1 i_2 \cdots i_l}$ of rank l which are invariant under rotations about an axis, say z-axis (azimuthal symmetry). One way to construct them is by taking the product $\hat{z}_{i_1}\hat{z}_{i_2}\cdots\hat{z}_{i_l}$, where $\hat{z}$ is the unit-vector in the z-direction (i.e. $\hat{z}_i = \delta_{i3}$ is the i'th component of $\hat{z}$), and subtracting the trace to make the product traceless. Denote this symmetric traceless tensor by $S_{i_1 i_2 \cdots i_l} \equiv \{\hat{z}_{i_1}\hat{z}_{i_2}\cdots\hat{z}_{i_l}\}$. Show that the function

$$F_l(\hat{n}) \;=\; \{\hat{z}_{i_1}\hat{z}_{i_2}\cdots\hat{z}_{i_l}\}\hat{n}_{i_1}\hat{n}_{i_2}\cdots\hat{n}_{i_l} \tag{5}$$

   is nothing but the Legendre polynomial $P_l(\cos\theta)$ (upto some normalization constant). Here $\hat{n}_i$'s (with $i = 1, 2, 3$) are the components of the general unit vector

$$\hat{n}(\theta,\phi) \;=\; \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\,. \tag{6}$$

   (b) Now consider the general case when there is no azimuthal symmetry. Construct the symmetric traceless tensors and the function analogous to Eq. (7) (for $m > 0$):

$$F_l^m(\hat{n}) \;=\; \{\hat{u}^+_{i_1}\cdots\hat{u}^+_{i_m}\hat{z}_{i_{m+1}}\cdots\hat{z}_{i_l}\}\hat{n}_{i_1}\cdots\hat{n}_{i_l} \tag{7}$$

   where $\hat{u}^{\pm} = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$. Show that $F_l^m(\hat{n})$'s are nothing but the spherical harmonics $Y_l^m(\theta,\phi)$, again upto a normalization constant.

This neat connection between tensors and spherical harmonics can be understood from the general solution to *Laplace's equation*, which is the underlying basis for multipole expansion, a widely used technique in physics.

3. **Creating states from vacuum:** In class, we discussed the Hermitian number operator $N \equiv a^\dagger a$ (where $a, a^\dagger$ are the creation and annihilation operators, respectively) and its eigenvectors $|n\rangle$ with non-negative integer eigenvalues n.

   (a) Show that

$$|n\rangle \;=\; \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \tag{8}$$

   where $|0\rangle$ is the ground (vacuum) state such that $a|0\rangle = 0$. Eq. (8) is very useful in quantum mechanics, and we have already seen one application in the *Jordan-Schwinger construction* of the angular momentum algebra.

   (b) Using Eq. (8), show that $[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$.