PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

Bhupal Dev **Homework 5** Due: 02/23/17

1. **Jacobi Identity:**

   (a) Show that X, Y, Z being three matrices (or operators or generators)

   $$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] \;=\; 0\,. \tag{1}$$

   This is the *Jacobi identity*, which often comes in handy while dealing with Lie algebra,[1] and sometimes even used as part of the definition of Lie algebra.

   (b) (Bonus Question) The geometrical meaning of Jacobi identity is contained in a simple statement you learned in high-school math, namely, the three altitudes of a triangle intersect at one point. Prove it (if you can).

2. **Determinant and Levi-Civita:**

   (a) Show that the determinant of an n-by-n matrix M can be written compactly in terms of the totally antisymmetric Levi-Civita tensor, as follows:[2]

   $$\varepsilon^{pqr\cdots s}\det M \;=\; \varepsilon^{ijk\cdots n}M^{ip}M^{jq}M^{kr}\cdots M^{ns}\,. \tag{2}$$

   We used this in class to show that $\varepsilon^{ijk\cdots n}$ is an invariant tensor under rotation.

   (b) Using Eq. (2), show that for the rotation matrix $R \in SO(N)$, we have

   $$\varepsilon^{ijk\cdots n}R^{ip}R^{jq} \;=\; \varepsilon^{pqr\cdots s}R^{kr}\cdots R^{ns}\,. \tag{3}$$

3. $SO(3)$ **is Special:**

   (a) If A^{ijk} is a totally antisymmetric tensor of rank-3, where $i,j,k = 1,2,\cdots,N$, then show that A has $N(N-1)(N-2)/3!$ components.

   (b) For $N=3$, identify the one component of A. Show that it transforms as a *scalar* (i.e. remains the same) under rotation. This, together with the result we derived in class for the symmetric tensors, makes $SO(3)$ special among the $SO(N)$ groups.

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[1] In vector notation, Eq. (1) might look more familiar: $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$.

[2] Remember that we are using the *Einstein summation convention*, i.e. repeated indices are always summed over, unless otherwise specified.