PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Young Tableaux:**[1] We have seen that the Young tableaux with n boxes are in one-to-one correspondence with the irreps of S_n. Use this information to decompose the permutation group S_5 into its irreps [2] and verify that they satisfy the general formula

$$N(G) \;=\; \sum_r d_r^2\,, \tag{1}$$

where $N(G)$ is the order of the group and d_r is the dimension of the irrep r. This should tell you that the decomposition in Eq. (1) is unique for permutation groups (and hence, for all finite groups, by Cayley's theorem).

2. **Exponential of Matrices:** The results you will derive below are important for the *Lie algebra* (our subject of discussion for the next several classes).

   (a) For two matrices X and Y, if $[X,Y]$ commutes with both X and Y (such a commutator $[X,Y]$ is known as a *central commutator*), show that

$$e^{\alpha X} Y e^{-\alpha X} \;=\; Y + \alpha[X,Y]\,, \tag{2}$$

   where α is some arbitrary complex number.

   (b) Using Eq. (2), show that[3]

$$e^X e^Y \;=\; e^{X+Y+\frac{1}{2}[X,Y]}\,. \tag{3}$$

   As an immediate consequence of Eq. (3), show that

$$\mathrm{Tr}\,\log(e^X e^Y) \;=\; \mathrm{Tr}\,X + \mathrm{Tr}\,Y\,. \tag{4}$$

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[1] Young Tableaux have numerous applications in representation theory, combinatorics, and algebraic geometry. We will need them later for finding the irreps and direct products of Lie groups. So make friendship with them!

[2] *Hint:* You can use the *hook length formula* we discussed in class.

[3] *Hint:* An easy way is to take the derivative of the function $e^{\alpha X}e^{\alpha Y}$ w.r.t. α and set $\alpha = 1$ at the end.

(c) Derive the more general form of Eq. (2), i.e. for non-central $[X, Y]$, show that[4]

$$e^{\alpha X} Y e^{-\alpha X} \;=\; Y + \alpha[X,Y] + \frac{\alpha^2}{2!}[X,[X,Y]] + \cdots + \frac{\alpha^n}{n!}[X,[X,\cdots[X,Y]\cdots]] + \cdots\,, \tag{5}$$

which is just an infinite series of nested commutators (and nothing else!). This non-trivial result is known as the *Baker-Campbell-Hausdorff formula* and has many applications in physics, including quantum mechanics, lattice gauge theory and, of course, Lie algebra.

3. **Structure Constants:** For a Lie group, the commutator between any two generators can be written as a linear combination of the generators, i.e.

$$[T_a, T_b] \;=\; i f_{abc} T_c\,. \tag{6}$$

Show that for Hermitian generators,[5] the coefficients f_{abc} must be real. These are known as the *structure constants* of the corresponding Lie algebra, which essentially characterize the Lie group.

[4] *Hint:* You can use a similar trick as in part (b) by differentiating a function $e^{\lambda\alpha X} Y e^{-\lambda\alpha X}$ w.r.t. λ and then setting $\lambda = 1$ at the end.

[5] We can always construct a Hermitian matrix T from a real, anti-symmetric matrix A by taking $T = -iA$.