PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Symplectic Groups:**

   (a) Prove that a $2n \times 2n$ matrix R satisfying the symplectic condition $R^T J R = J$ has determinant $+1$. Here $J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$, as we discussed.

   (b) Show that the generator of the symplectic group, given by the $2n \times 2n$ Hermitian matrix H satisfying the condition $H^T = JHJ$, can be written as a linear combination of the Hermitian traceless matrices $iA \otimes \mathbb{1}$ and $S_i \otimes \sigma_i$, where A is an arbitrary real $n \times n$ antisymmetric matrix, S_i (with $i = 1, 2, 3$) are arbitrary real $n \times n$ symmetric matrices and σ_i are the usual 2×2 Pauli matrices.

2. **Chevalley Basis for $SU(3)$:** We can get rid of the pesky square root factors appearing in the roots of $SU(N)$ by going to what's called the Chevalley basis. Let's see this explicitly for $SU(3)$.

   (a) Replace the last Gell-Mann matrix λ_8 with $h^2 = \text{diag}(0, 1, -1)$. Show that h^2, together with the appropriate raising and lowering matrices (calculate and call them e^2 and $(e^2)^T$) in the 2–3 sector, forms an $SU(2)$ algebra. Do the same in the 1–2 sector with $h^1 = \text{diag}(1, -1, 0)$, which is same as λ_3. Call the corresponding raising and lowering matrices as e^1 and $(e^1)^T$. Show that e^1 and e^2 represent the two simple roots of $SU(3)$ (without the factors of square root this time!).

   (b) Show that $e^3 = [e^1, e^2]$ gives the third positive root. Why can't we have more positive roots by taking double commutators, such as $[e^1, [e^1, e^2]]$ or $[e^2, [e^1, e^2]]$?

   (c) From the above exercise, you can (hopefully) convince yourself that the eight matrices $h^1, h^2, e^1, (e^1)^T, e^2, (e^2)^T, e^3, (e^3)^T$ (the 'new' Gell-Mann matrices) generate the $SU(3)$ algebra. But it comes with a price, i.e. $\text{Tr}(h^i h^j) \neq \delta^{ij}$. Can you tell why it might be troublesome? (*Hint:* Think in terms of a metric.)