PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

Bhupal Dev **Homework 10** Due: 04/13/17

1. **Gell-Mann-Okubo Mass formula:** The meson octet $(\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$ is known to populate the adjoint representation **8** of $SU(3)$, given by a traceless tensor Φ^i_j (with $i, j = 1, 2, 3$) which in the matrix form looks like

$$\Phi \; = \; \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \phi_a \lambda_a \; = \; \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \qquad (1)$$

where λ_a's are the Gell-Mann matrices.

   (a) Identify the components ϕ_a's from Eq. (1).

   (b) Calculate $\mathrm{Tr}(\Phi^2)$. Given that the mass term in the Lagrangian is $-(1/2)\mathrm{Tr}(m_0^2\Phi^2)$, what can you tell about the masses of the eight mesons with a perfect $SU(3)$ symmetry?

   (c) Show that the mass term written above belongs to the representation $\mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1}$.

   (d) Experimentally, we know that the masses don't follow the relation you got in part (b). So we have to break the $SU(3)$ symmetry, which can be done by either **27** or **8** from part (c).[1] It turns out that the right answer is **8**.[2] Calculate the breaking term $\mathrm{Tr}(\Phi^2\lambda_8)$.

   (e) Using the results from part (d), show that after the $SU(3)$-symmetry breaking, we obtain the following relation among the meson masses:

$$4m_K^2 \; = \; 3m_\eta^2 + m_\pi^2 \,, \qquad (2)$$

   famously known as the Gell-Mann-Okubo mass formula (which works well!).

2. **Roots of $SU(4)$:** Calculate all the root vectors for $SU(4)$ using the $SU(2)$ subalgebra method and verify that the ones derived in class (using the weight method) are indeed the simple roots.

---

[1] Why not **1**?

[2] Trust the intuition of Feynman and Gell-Mann (if you don't know enough about quarks).