PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Isomorphism of cyclic groups:**

   (a) Show that $Z_2 \otimes Z_4 \neq Z_8$, but $Z_2 \otimes Z_5 = Z_{10}$.

   (b) This is a general phenomenon, i.e. $Z_p \otimes Z_q$ is isomorphic to $Z_{p\times q}$, as long as p and q are *relatively* prime. Note that p, q are not necessarily prime numbers themselves. Can you check this explicitly by taking the example of $Z_3 \otimes Z_4$?

2. **Hurwitz algebra:** This is defined by the *norm property*, i.e. the norm of the product of any two elements a and a' is the product of their norms:

$$N(a\, a') \;=\; N(a)\; N(a')\,, \tag{1}$$

where the norm is defined in the usual manner, $N(a) \equiv \sqrt{a\bar{a}}$, with $\bar{a}$ being the conjugate of a. Surprisingly, there are *only* four algebras satisfying Eq. (1). Two of them are very familiar to you: the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$. Here are the other two: the *quaternions* $\mathbb{Q}$ with three imaginary units, and the *octonions* $\mathbb{O}$ with seven imaginary units.

   (a) A quaternion q and its conjugate $\bar{q}$ are defined as

$$q \;\equiv\; x_0 + \sum_{i=1}^{3} e_i x_i\,, \qquad \bar{q} \;\equiv\; x_0 - \sum_{i=1}^{3} e_i x_i\,, \tag{2}$$

   where x_0, x_i are real numbers and e_i's are the imaginary units. Show that for any two quaternions q and q', the norm property (1) is satisfied if and only if

$$e_i e_j \;=\; -\delta_{ij} + \epsilon_{ijk} e_k\,, \tag{3}$$

   where δ_{ij} and ϵ_{ijk} are the usual Kronecker delta and Levi-Civita tensor, respectively. Eq. (3) is sometimes known as *Hamilton's rule*. Do you see any similarity between the e_i's and the good-old Pauli matrices?

(b) (Bonus question) An octonion o and its conjugate $\bar{o}$ are defined similar to Eq. (2):

$$o \equiv x_0 + \sum_{i=1}^{7} e_i x_i \,, \qquad \bar{o} \equiv x_0 - \sum_{i=1}^{7} e_i x_i \,, \tag{4}$$

where x_0, x_i are real numbers and e_i's are the imaginary units. You can repeat the exercise as above to convince yourself that for any two octonions o and o', the norm property (1) is satisfied if and only if the e_i's satisfy

$$e_i e_j = -\delta_{ij} + \psi_{ijk} e_k \,, \tag{5}$$

where ψ_{ijk} are the totally antisymmetric *octonion structure functions*, whose only non-zero elements are $\psi_{123} = \psi_{246} = \psi_{435} = \psi_{651} = \psi_{572} = \psi_{714} = \psi_{367} = 1$. Eq. (5) is sometimes known as the *Cayley algebra*.

We will discuss later (in class) the matrix representations of both quaternions and octonions in terms of $SO(N)$ algebra. This has important physics applications, e.g. in string theory.

3. **Cycle structure of the permutation group:** Show that the number of elements in a permutation group S_n with a given cycle structure is given by

$$\frac{n!}{\Pi_{j=1}^{k} j^{n_j} n_j!} \,, \tag{6}$$

where n_j is the number of j-cycles in the cycle structure. Remember that $n = \sum_{j=1}^{k} j n_j$, where k is the cycle of maximum length in S_n. As an example, list all possible cycle structures in S_5 and count the number of elements with each structure using Eq. (6).