PHYSICS 590 II: GROUP THEORY AND SYMMETRIES IN PHYSICS

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1. **Reality of $SU(2)$ representations:**

   (a) For any 2×2 unitary matrix U with unit determinant, show that there exists a matrix S such that $S^{-1}US = U^*$.

   (b) For $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in the fundamental representation **2** of $SU(2)$ with eigenvalues $\pm\frac{1}{2}$ for the diagonal generator T_3, i.e.

   $$T_3\psi_1 = \frac{1}{2}\psi_1\,, \quad T_3\psi_2 = -\frac{1}{2}\psi_2\,, \tag{1}$$

   calculate the corresponding eigenvalues of the T_3 generator for the conjugate representation $\mathbf{2}^*$.

2. $SU(2)$ **subalgebra of** $SU(3)$ **in terms of fields:** Given the quark field operators $\begin{pmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{pmatrix}$ in the fundamental representation **3** of $SU(3)$, satisfying the anticommutation relations

   $$\{q_a(x), q_b^\dagger(y)\} = \delta_{ab}\delta^3(x-y)\,, \tag{2}$$

   (a) calculate the commutators $[W_a^b, W_c^d]$ for the non-Hermitian generators

   $$W_a^b = \int d^3x\; q_b^\dagger(x)q_a(x)\,. \tag{3}$$

   (b) Show that W_2^1, W_3^2 and W_3^1 are nothing but the I-spin, U-spin and V-spin raising operators, respectively.

   (c) In this notation, calculate the third component of the isospin and the hypercharge operators in terms of W_a^b.

3. **Isospin breaking effects:** The $SU(2)$ isospin symmetry is broken in nature by electromagnetic interactions (involving the emission and absorption of a photon), containing both $\Delta I = 1$ and $\Delta I = 2$ terms, as well as by strong interactions (due to

mass difference between up and down-type quarks), containing only $\Delta I = 1$ piece. Just using the Wigner-Eckart theorem (and Clebsch-Gordan coefficients), calculate the relative mass splitting for the $I = \frac{3}{2}$ isomultiplet $(\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-})$ due to the isospin-changing $\Delta I = 1$ and $\Delta I = 2$ effects, and show the mass relation

$$m_{\Delta^{++}} - m_{\Delta^{-}} = 3(m_{\Delta^{+}} - m_{\Delta^{0}}) \,. \tag{4}$$

4. **Lorentz and Poincaré algebra:** Using the definition of the Lorentz tensor

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{5}$$

and the fact that $\partial_\mu x_\nu = \eta_{\mu\nu}$, where $\eta = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric in (3+1)-dimensions,

(a) derive the commutation relation

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}) \,. \tag{6}$$

(b) The Lorentz algebra (6) is extended to the Poincaré algebra by supplementing the generators $J_{\mu\nu}$ (of rotations and boosts) by the generators of translation $P_\mu = i\partial_\mu$. Derive the additional commutation relations

$$[J_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \tag{7}$$

$$[P_\mu, P_\nu] = 0 \,. \tag{8}$$

(c) Given the Pauli-Lubanski vector

$$W_\sigma \equiv -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} J^{\mu\nu} P^\rho \,, \tag{9}$$

derive the commutation relation

$$[W_\mu, W_\nu] = i\varepsilon_{\mu\nu\rho\sigma} W^\rho P^\sigma \,. \tag{10}$$