PHYSICS 543: GROUP THEORY AND SYMMETRIES IN PHYSICS

Homework 9 Due: April 5, 2019

1. **Electric Quadrupole Transition:** The components of the electric quadrupole operator $\boldsymbol{D}$ are defined as

$$D_{ij} \;=\; 3r_ir_j - r^2\delta_{ij}\,, \tag{1}$$

where r_i (with $i = 1, 2, 3$) are the spatial (x, y, z) coordinates.

   (a) Show that there are only five independent components of $\boldsymbol{D}$, which transform under $SO(3)$ rotation as a $j = 2$ representation.

   (b) For a single electron transition from one atomic orbital labeled by $|n, j, m\rangle$ to another labeled by $|n', j', m'\rangle$ (which are simultaneous eigenstates of J_z and $\boldsymbol{J}^2$, with J_i's being the angular momentum operators) due to the application of $\boldsymbol{D}$, what values of $\Delta j = j' - j$ and $\Delta m = m' - m$ are allowed?

2. **Finding $\boldsymbol{SU(2)}$ Group Elements:** We argued that any 2×2 special unitary matrix can be written as a linear combination of the Pauli matrices (the generators for $SU(2)$). Let us demonstrate it in another way:

   (a) Start with an arbitrary *complex* 2×2 matrix

$$M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2}$$

   Show that the special unitarity conditions ($M^\dagger M = \mathbf{1}$ and det $M = 1$) reduce the number of *real* parameters from 8 to just 3.

   (b) Call them (r, ϕ, φ). Show that M can be written as

$$M \;=\; \begin{pmatrix} \sqrt{1-r^2}\; e^{i\phi} & re^{i\varphi} \\ -re^{-i\varphi} & \sqrt{1-r^2}\; e^{-i\phi} \end{pmatrix}. \tag{3}$$

(c) Show that Eq. (3) is the same as the more familiar $SU(2)$ group element in terms of the generators:

$$U = e^{i\theta\hat{n}\cdot\vec{\sigma}/2}, \tag{4}$$

where σ_i's (with $i = 1, 2, 3$) are the Pauli matrices. Identify θ and $\hat{n}$ in terms of r, ϕ, φ.

3. **Fun with Pauli and Gell-Mann:**

(a) Using the Pauli and Gell-Mann matrices, calculate the structure constants for $SU(2)$ and $SU(3)$, respectively.

(b) Show that the Pauli and Gell-Mann matrices respectively satisfy the anti-commutation relations

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}, \tag{5}$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} + 2d_{abc}\lambda_c, \tag{6}$$

where $j, k = 1, 2, 3$, and $a, b, c, = 1, 2, \cdots, 8$ and the d-symbols in Eq. (6) are a bunch of *real* numbers (evaluate them!).

Can you tell the main difference between these d_{abc} and the $SU(3)$ structure constants f_{abc} (apart from the fact that they are not identical)?

(c) Using the above results, verify the statement that all $SU(N)$ generators T_a (with $a = 1, 2, \cdots, N^2 - 1$) can be normalized by

$$\mathrm{Tr}(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{7}$$

for $SU(2)$ and $SU(3)$.