PHYSICS 543: GROUP THEORY AND SYMMETRIES IN PHYSICS

Homework 8 Due: March 29, 2019

1. **Legendre polynomials:** Use the tensor approach to work out $P_5(\cos\theta)$.

2. **Creating states from vacuum:** Given the Hermitian number operator $N \equiv a^\dagger a$ (where $a^\dagger, a$ are the creation and annihilation operators, respectively) and its eigenvectors $|n\rangle$ with non-negative integer eigenvalues n,

   (a) Show that

   $$|n\rangle \;=\; \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle\,, \tag{1}$$

   where $|0\rangle$ is the ground (vacuum) state such that $a|0\rangle = 0$. Eq. (1) is very useful in quantum mechanics.

   (b) Using Eq. (1), show that $[a,(a^\dagger)^n] = n(a^\dagger)^{n-1}$.

3. **Clebsch-Gordan decomposition:** Using the Clebsch-Gordan (CG) decomposition of $j \otimes j'$ in the form

   $$|J, M\rangle \;=\; \sum_{m=-j}^{j}\sum_{m'=-j'}^{j'} |j, j', m, m'\rangle\langle j, j', m, m'|J, M\rangle\,, \tag{2}$$

   (a) Work out the CG coefficients for $j = 1$ and $j' = \frac{1}{2}$ and show that

   $$1 \otimes \frac{1}{2} \;=\; \frac{3}{2} \oplus \frac{1}{2}\,. \tag{3}$$

   What does it mean in terms of the dimensions of the corresponding irreps?

   (b) Work out the CG coefficients for $j = 2$ and $j' = 1$ and show that

   $$2 \otimes 1 \;=\; 3 \oplus 2 \oplus 1\,. \tag{4}$$

   What does it mean in terms of the dimensions of the corresponding irreps?