PHYSICS 543: GROUP THEORY AND SYMMETRIES IN PHYSICS

Homework 7 Due: March 22, 2019

1. ***SO*(3) is special:**

   (a) If A^{ijk} is a totally antisymmetric tensor of rank-3, where $i, j, k = 1, 2, \cdots, N$, then show that A has $N(N-1)(N-2)/3!$ components.

   (b) For $N = 3$, identify the one component of A. Show that it transforms as a *scalar* (i.e. remains the same) under rotation. This, together with the result we derived in class for the symmetric tensors, makes $SO(3)$ special among the $SO(N)$ groups.

2. **Spherical harmonics:** This warm-up exercise is to refresh your memory on orbital angular momentum operators in quantum mechanics. [*Hint:* If have never seen this stuff before, please refer to any undergrad quantum mechanics textbook, e.g. D. J. Griffiths, *Introduction to Quantum Mechanics*, Prentice Hall (1995).]

   (a) Applying the Heisenberg prescription $\vec{p} \to -i\hbar\,\vec{\nabla}$ to the classical angular momentum $\vec{L} = \vec{x} \times \vec{p}$, we can write it as a differential operator: $\vec{L} = -i\hbar\,\vec{x} \times \vec{\nabla}$. Write down the explicit forms of L_z, $L_\pm = L_x \pm iL_y$ and $\vec{L}^2$ in spherical polar coordinates (r, θ, ϕ).

   (b) The eigenvalue equations for L_z and $\vec{L}^2$ can be written in terms of the *spherical harmonics* $Y_l^m(\theta, \phi) = \langle \theta, \phi \mid l, m \rangle$:

$$L_z Y_l^m(\theta,\phi) \;=\; m\hbar\, Y_l^m(\theta,\phi)\,, \tag{1}$$

$$\vec{L}^2 Y_l^m(\theta,\phi) \;=\; l(l+1)\hbar^2\, Y_l^m(\theta,\phi)\,, \tag{2}$$

   where $l = 0, 1, 2, \cdots$ is a non-negative integer and $m = -l, -l+1, \cdots, l-1, l$. Using Eq. (1) and the results from part (a), show that the appropriately normalized eigenfunctions $Y_l^m(\theta, \phi)$ are given by

$$Y_l^m(\theta,\phi) \;=\; (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi} \tag{3}$$

for $m \geq 0$ and $Y_l^{-m} = (-1)^m (Y_l^m)^*$. Here $P_l^0(\cos\theta) \equiv P_l(\cos\theta)$ are the *Legendre polynomials* and $P_l^m(\cos\theta)$ are the *associated* Legendre polynomials, given by (for $-l \leq m \leq l$)

$$P_l^m(u) \;=\; \frac{(1-u^2)^{m/2}}{2^l l!} \left(\frac{d}{du}\right)^{l+m} (u^2-1)^l \,. \tag{4}$$

You might find it useful (for the next problem) to list a few of these explicitly!

[*Hint:* The solution of this problem involves several steps: (i) Use separation of variables and operate by L_z to extract the ϕ dependence; (ii) use $L_\pm Y_l^{\pm l}(\theta,\phi) = 0$ to extract the θ dependence; (iii) use the normalization condition $\langle Y_l^m | Y_l^m \rangle = 1$ to fix the integration constant.]

3. **Symmetric traceless tensors**

(a) Consider the symmetric traceless tensors $S_{i_1 i_2 \cdots i_l}$ of rank l which are invariant under rotations about an axis, say z-axis (azimuthal symmetry). One way to construct them is by taking the product $\hat{z}_{i_1} \hat{z}_{i_2} \cdots \hat{z}_{i_l}$, where $\hat{z}$ is the unit-vector in the z-direction (i.e. $\hat{z}_i = \delta_{i3}$ is the ith component of $\hat{z}$), and subtracting the trace to make the product traceless. Denote this symmetric traceless tensor by $S_{i_1 i_2 \cdots i_l} \equiv \{\hat{z}_{i_1} \hat{z}_{i_2} \cdots \hat{z}_{i_l}\}$. Show that the function

$$F_l(\hat{n}) \;=\; \{\hat{z}_{i_1} \hat{z}_{i_2} \cdots \hat{z}_{i_l}\} \hat{n}_{i_1} \hat{n}_{i_2} \cdots \hat{n}_{i_l} \tag{5}$$

is nothing but the Legendre polynomial $P_l(\cos\theta)$ (upto some normalization constant). Here $\hat{n}_i$'s (with $i = 1, 2, 3$) are the components of the general unit vector

$$\hat{n}(\theta,\phi) \;=\; \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}\,. \tag{6}$$

(b) Now consider the general case when there is no azimuthal symmetry. Construct the symmetric traceless tensors and the function analogous to Eq. (7) (for $m > 0$):

$$F_l^m(\hat{n}) \;=\; \{\hat{u}^+_{i_1} \cdots \hat{u}^+_{i_m} \hat{z}_{i_{m+1}} \cdots \hat{z}_{i_l}\} \hat{n}_{i_1} \cdots \hat{n}_{i_l} \tag{7}$$

where $\hat{u}^\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$. Show that $F_l^m(\hat{n})$'s are nothing but the spherical harmonics $Y_l^m(\theta,\phi)$, again upto a normalization constant.

This neat connection between tensors and spherical harmonics can be understood from the general solution to *Laplace's equation*, which is the underlying basis for multipole expansion, a widely used technique in physics.