PHYSICS 543: GROUP THEORY AND SYMMETRIES IN PHYSICS

Homework 10 Due: April 12, 2019

1. **Bosonic harmonic oscillator:** The $SU(2)$ Lie algebra satisfies an interesting property (called the *involutive automorphism*) that we can reverse the sign of two generators, e.g. $T_{1,2} \to -T_{1,2}$, $T_3 \to T_3$, still preserving the algebra. We will use this property to obtain a closely related algebra.

   (a) Consider the new generators $L_{1,2} = iT_{1,2}$, $L_3 = T_3$. Work out the commutation relations $[L_i, L_j]$ and show that they form a Lie algebra, i.e. they satisfy the Jacobi identity.

   (b) What is the Casimir operator? Does the form look something familiar? What can you tell about the algebra from this?

   (c) From the answer to part (b), you can presumably guess that the representation theory of this algebra is much different from the $SU(2)$ algebra (even a sign change matters!). It turns out that the simplest unitary representation is in the infinite Hilbert space generated by a *bosonic* harmonic oscillator:

   $$L_+ = \frac{1}{2\sqrt{2}} a^\dagger a^\dagger \,, \quad L_- = \frac{1}{2\sqrt{2}} aa \,, \quad L_3 = \frac{1}{4}(1 + 2a^\dagger a) \,, \tag{1}$$

   where $L_\pm \equiv (L_1 \pm iL_2)/\sqrt{2}$ and $a, a^\dagger$ are the annihilation and creation operators, respectively. Check that they satisfy the algebra you derived in part (a). Show that the L_3 spectrum is bounded from below and there are *two* infinite towers of states (corresponding to the even and odd subsets of the algebra).

2. **Isospin symmetry:** Using isospin symmetry, predict the ratio of the cross sections for $p + {}^2\text{H} \to {}^3\text{H} + \pi^+$ and $p + {}^2\text{H} \to {}^3\text{He} + \pi^0$. You can use the fact that the ^{3}H (tritium, a *pnn* bound state) and ^{3}He (isotope of helium, a *ppn* bound state) are known to form an isospin 1/2 doublet, whereas ^{2}H (deuteron, a *pn* bound state) has isospin 0.