PHYSICS 543: GROUP THEORY AND SYMMETRIES IN PHYSICS

Homework 1 Due: January 25, 2019

1. **Isomorphism of cyclic groups:**

   (a) Show that $Z_2 \otimes Z_4 \neq Z_8$.

   (b) Show that $Z_2 \otimes Z_5 = Z_{10}$.

   (c) Generalize these observations to show that $Z_p \otimes Z_q$ is isomorphic to $Z_{p\times q}$, as long as p and q are *relatively* prime (p, q are not necessarily prime numbers themselves).

2. **Quaternion group:**

   (a) List all possible groups of order 8, specifying their presentations and ranks.

   (b) From your answer to part (a), you should have found a new group Q of rank 2 with the presentation

$$Q: \ \langle A, B \mid A^4 = B^4 = I, \ A^2 = B^2, \ BAB^{-1} = A^{-1} \rangle \, . \tag{1}$$

   Show that the generators of Q can be written in terms of the quaternions. That's why Q is called the *quaternion group*.

   Hint: Quaternions are generalizations of complex numbers, with three imaginary units (instead of one $i = \sqrt{-1}$):

$$q \ \equiv \ x_0 + \sum_{i=1}^{3} e_i x_i \, , \qquad \bar{q} \ \equiv \ x_0 - \sum_{i=1}^{3} e_i x_i \, , \tag{2}$$

   where x_0, x_i are real numbers and e_i's are the imaginary units satisfying Hamilton's multiplication rules:

$$e_i e_j \ = \ -\delta_{ij} + \epsilon_{ijk} e_k \, , \tag{3}$$

   where δ_{ij} and ϵ_{ijk} are the usual Kronecker delta and Levi-Civita tensor, respectively. Do you see any similarity between the e_i's and the good-old Pauli matrices?

   We will discuss later the matrix representations of quaternions in terms of the $SO(N)$ algebra. This has important physics applications, e.g. in string theory.