1. [5 points] For each of the following reactions, indicate what kind of interaction (Strong, Electromagnetic, Weak, or None) is responsible and why: (a) $\pi^0 \rightarrow \gamma + \gamma$, (b) $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$, (c) $\Lambda \rightarrow p + \pi^-$, (d) $\Delta^0 \rightarrow p + \pi^-$, (e) $p \rightarrow e^+ + \gamma$.

(a) $\pi^0 \rightarrow \gamma \gamma$ is via [electromagnetic] interaction, since it has photons in the final state. Also, it has initial state $I = 1, I_3 = 0$ and final state $I = 0$. So conserves $I_3$, but not $I$.

(b) $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ is via [weak] interaction, since it contains neutrino final state. It takes $I = 1, I_3 = -1$ to $I = 0$, so doesn’t conserve either $I$ or $I_3$.

(c) $\Lambda (uds) \rightarrow p + \pi^-$ takes $S = -1$ to $S = 0$, so doesn’t conserve strangeness. hence, must be via [weak] interaction.

(d) $\Delta^0 \rightarrow p + \pi^-$ conserves strangeness. Also $I = 3/2$ in the initial state and $I = 1/2 + 1 = 3/2$ in the final state. Similarly, $I_3 = -1/2$ in the initial state and $I_3 = 1/2 - 1 = -1/2$ in the final state. So conserves both $I$ and $I_3$. Must be [strong] interaction.

(e) $p \rightarrow e^+ + \gamma$ is [not allowed] in the Standard Model, because it violates baryon number: Initial state has $B = 1$ and final state has $B = 0$. This process must be absent (or highly suppressed, as in Grand Unified Theories) to ensure proton stability (and our survival).

2. [5 points] What is the probability of a muon (with rest mean lifetime of $2.2 \times 10^{-6}$s) lasting more than 1 second in its rest frame?

The probability of decay at any given time $t$ is simply $e^{-\Gamma t} = e^{-t/\tau}$, where $\tau = 2.2 \times 10^{-6}$s is the mean lifetime. So for $t = 1$ s, we have the probability $e^{-1/(2.2 \times 10^{-6})} = e^{-4.55 \times 10^5}$ (pretty small!). However, due to the time dilation effect, they can live much longer in the lab frame.
3. (a) [10 points] Using the meson mass formula

\[ M_{\text{meson}} = m_1 + m_2 + A \frac{S_1 \cdot S_2}{m_1 m_2}, \]  

(1)
calculate the mass splitting between the \( \pi \) and \( \rho \) mesons. Use \( m_u = m_d = 308 \text{ MeV}/c^2 \) for the constituent quark masses and \( A = (2m_u/\hbar)^2 \times 159 \text{ MeV}/c^2 \) for the constant in Eq. (1).

We have

\[ S_1 \cdot S_2 = \frac{1}{2}(S^2 - S_1^2 - S_2^2) = \frac{1}{2} [s(s + 1) - s_1(s_1 + 1) - s_2(s_2 + 1)] \hbar^2 \]

\[ = \begin{cases} 
- \frac{3}{4} \hbar^2 & \text{for pseudoscalar } (s = 0) \\
\frac{1}{4} \hbar^2 & \text{for vector } (s = 1) 
\end{cases}. \]  

(2)

So for pseudoscalar pions, Eq. (1) becomes

\[ M_\pi = 2m_u + \frac{A}{m_u^2} \left( - \frac{3}{4} \hbar^2 \right), \]  

(3)
and for the vector rho mesons, Eq. (1) becomes

\[ M_\rho = 2m_u + \frac{A}{m_u^2} \left( \frac{1}{4} \hbar^2 \right). \]  

(4)

So the mass splitting is given by

\[ \Delta M \equiv M_\rho - M_\pi = \frac{A}{m_u^2} \hbar^2 = 4 \times 159 \text{ MeV}/c^2 = \boxed{636 \text{ MeV}/c^2}. \]  

(5)

(b) [10 points] Using the baryon mass formula

\[ M_{\text{baryon}} = m_1 + m_2 + m_3 + A' \left[ \frac{S_1 \cdot S_2}{m_1 m_2} + \frac{S_2 \cdot S_3}{m_2 m_3} + \frac{S_1 \cdot S_3}{m_1 m_3} \right], \]  

(6)
calculate the mass splitting between the \( \Delta \)-baryons and nucleons (proton/neutron). Use \( m_u = m_d = 363 \text{ MeV}/c^2 \) for the constituent quark masses and \( A' = (2m_u/\hbar)^2 \times 50 \text{ MeV}/c^2 \) for the constant in Eq. (6).

first note that the \( \Delta \)-baryons are in a decuplet with \( j = 3/2 \) and the nucleons are in an octet with \( j = 1/2 \). With \( L = 0 \), we have

\[ J^2 = (S_1 + S_2 + S_3)^2 = S_1^2 + S_2^2 + S_3^2 + 2(S_1 \cdot S_2 + S_1 \cdot S_3 + S_2 \cdot S_3), \]  

(7)
with eigenvalue of \( j(j + 1)\hbar^2 = \frac{15}{4} \hbar^2 \) for \( j = \frac{3}{2} \) (decuplet) and \( \frac{3}{4} \hbar^2 \) for \( j = \frac{1}{2} \) (octet).

So from Eq. (7),

\[
S_1 \cdot S_2 + S_1 \cdot S_3 + S_2 \cdot S_3 = \frac{\hbar^2}{2} \left[ j(j + 1) - 3 \cdot \left( \frac{1}{2} + 1 \right) \right] = \begin{cases} 
\frac{3}{4} \hbar^2 & \text{for } j = \frac{3}{2} \\
-\frac{3}{4} \hbar^2 & \text{for } j = \frac{1}{2}.
\end{cases}
\]

(8)

When the three quark masses are equal, i.e. for the baryons \( \Delta \) (decuplet) and \( N \) (octet) entirely made up of \( u \) and \( d \) combinations, we can simply use Eq. (8) in Eq. (6) to get

\[
M_\Delta = 3m_u + \frac{3\hbar^2}{4m_u^2}A'.
\]

(9)

\[
M_N = 3m_u - \frac{3\hbar^2}{4m_u^2}A'.
\]

(10)

So the mass splitting is given by

\[
\Delta M \equiv M_\Delta - M_N = \frac{3\hbar^2}{2m_u^2}A' = \frac{3}{2} \times 4 \times 50 \text{ MeV}/c^2 = \boxed{300 \text{ MeV}/c^2}.
\]

(11)

4. [10 points] A Uranium-238 nucleus at rest undergoes alpha-decay (by emission of an alpha-particle, i.e. Helium-4) to Thorium-234. Find the energy and momentum of the alpha particle in terms of its mass and the masses of the Uranium and Thorium nuclei.

From four-momentum conservation, \( p_U^\mu = p_{Th}^\mu + p_\alpha^\mu \), or \( p_{Th}^\mu = p_U^\mu - p_\alpha^\mu \). Taking the scalar product of each side with itself, we get

\[
p_{Th}^2 = p_U^2 + p_\alpha^2 - 2p_U \cdot p_\alpha.
\]

(12)

Since \( p^2 = m^2c^2 \), we have \( p_U^2 = m_U^2c^2 \), and similarly for \( p_{Th} \) and \( p_\alpha \). Also in the rest frame of \( ^{238}\text{U} \), \( p_U = (m_Uc, 0) \) and \( p_\alpha = \left( \frac{E_\alpha}{c}, \mathbf{p}_\alpha \right) \). So \( p_U \cdot p_\alpha = m_U E_\alpha \). Substituting these into Eq. (12), we get

\[
m_{Th}^2c^2 = m_U^2c^2 + m_\alpha^2c^2 - 2m_U E_\alpha,
\]

or,

\[
E_\alpha = \frac{m_U^2 - m_{Th}^2 + m_\alpha^2}{2m_U}c^2.
\]

(13)
As for the three-momentum, \(|\mathbf{p}_{\text{Th}}| = |\mathbf{p}_o|\) due to momentum conservation in the rest frame of \(^{238}\text{U}\). So we can use the energy-momentum relation to get:

\[
E_o^2 = |\mathbf{p}_o|^2 c^2 + m_o^2 c^4,
\]

or,

\[
|\mathbf{p}_o|^2 = \frac{E_o^2}{c^2} - m_o^2 c^2 = \frac{(m_U^2 - m_{\text{Th}}^2 + m_o^2)^2 c^2}{4m_U^2} - m_o^2 c^2,
\]

or, \(|\mathbf{p}_o| = \frac{c}{2m_U} \sqrt{m_U^4 + m_{\text{Th}}^4 + m_o^4 - 2(m_U^2 m_{\text{Th}}^2 + m_U^2 m_o^2 + m_{\text{Th}}^2 m_o^2)}\)

or, \(|\mathbf{p}_o| = \frac{c}{2m_U} \lambda^{1/2}(m_U^2, m_{\text{Th}}^2, m_o^2)\) \(\text{(14)}\)

where \(\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)\) is the so-called triangle function.

5. \([10\text{ points}]\) Using isospin conservation, find the ratio of the rates for the strong decays \(\Sigma^0 \rightarrow K^- p\) and \(\Sigma^0 \rightarrow \bar{K}^0 n\).

Note that the \(\Sigma_0 : |1, 0\rangle\) is part of the \(I = 1\) baryon triplet, whereas \((p, n)\) and \((\bar{K}^0, K^-)\) both form isospin doublets, i.e. \(p : |\frac{1}{2}, \frac{1}{2}\rangle, n : |\frac{1}{2}, -\frac{1}{2}\rangle\), and similarly, \(\bar{K}^0 : |\frac{1}{2}, \frac{1}{2}\rangle, K^- : |\frac{1}{2}, -\frac{1}{2}\rangle\). Remember that the particle with higher electric charge should have higher \(I_3\) (in case you were not sure which one of the kaons should be \(I_3 = 1/2\)).

Now we use the Clebsch-Gordan decomposition for \(\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0\):

\[
|1, 1\rangle = \left| \frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2}\right\rangle,
\]

\[
|1, 0\rangle = \frac{1}{\sqrt{2}} \left[ \left| \frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2}\right\rangle + \left| \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2}\right\rangle \right],
\]

\[
|1, -1\rangle = \left| \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2}\right\rangle,
\]

\[
|0, 0\rangle = \frac{1}{\sqrt{2}} \left[ \left| \frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2}\right\rangle - \left| \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2}\right\rangle \right]. \tag{15}\]

So the wavefunctions for \((\bar{K}^0 n)\) and \((K^- p)\) states are given by

\[
\Psi(\bar{K}^0 n) = \left| \frac{1}{2}, \frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle + |0, 0\rangle)
\]

\[
\Psi(K^- p) = \left| \frac{1}{2}, -\frac{1}{2}\right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle - |0, 0\rangle). \tag{16}\]

Since \(\Sigma^0 \rightarrow K^- p\) and \(\Sigma^0 \rightarrow \bar{K}^0 n\) are both strong decays, \(I\) must be conserved, so the \(|0, 0\rangle\) part of the wavefunctions in Eq. (16) do not contribute. Moreover, the \(|I, I_3\rangle\) states are orthonormal to each other, so \(\langle I | H | I' \rangle = \mathcal{M}_I \delta_{II'}\), where \(H\) is the
Hamiltonian and $\mathcal{M}$ is the transition amplitude for $I = 1$. Thus, using Eqs. (16), the partial decay widths are given by

$$\Gamma_{\bar{K}^0 n} \propto |\langle \Psi(\Sigma^0)|H|\Psi(\bar{K}^0 n)\rangle|^2 = \frac{1}{2} |\langle 1, 0|H|1, 0\rangle|^2 \equiv \frac{|\mathcal{M}_1|^2}{2},$$

$$\Gamma_{K^- p} \propto |\langle \Psi(\Sigma^0)|H|\Psi(K^- p)\rangle|^2 = \frac{1}{2} |\langle 1, 0|H|1, 0\rangle|^2 \equiv \frac{|\mathcal{M}_1|^2}{2}.$$  

Hence we get

$$[\Gamma_{\bar{K}^0 n} : \Gamma_{K^- p} = 1 : 1].$$