PHYSICS 474: Introduction to Particle Physics

Homework 8 **Due: 11.30 03/18/2020**

1. **Dirac Hamiltonian and Conservation of Angular Momentum:**

   (a) [5 points] Using the Dirac equation, show that the corresponding Hamiltonian is given by

$$H = c\gamma^0(\boldsymbol{\gamma} \cdot \boldsymbol{p} + mc)\,. \tag{1}$$

   Hint: Solve the Dirac equation for cp^0 (total energy).

   (a) [5 points] Show that $[H, \boldsymbol{L}] = -i\hbar c\gamma^0(\boldsymbol{\gamma} \times \boldsymbol{p})$, where $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the orbital angular momentum.

   (b) [5 points] $[H, \boldsymbol{S}] = i\hbar c\gamma^0(\boldsymbol{\gamma} \times \boldsymbol{p})$, where

$$\boldsymbol{S} = \frac{\hbar}{2}\boldsymbol{\Sigma} \quad \text{with } \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\sigma} \end{pmatrix} \tag{2}$$

   is the spin angular momentum matrix for the Dirac particles.

   This result implies that $[H, \boldsymbol{J}] = 0$, where $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is the total angular momentum, i.e. $\boldsymbol{J}$ is a constant of motion, but not $\boldsymbol{L}$ or $\boldsymbol{S}$ separately.

2. **Dirac Spinors:** The four canonical solutions of the Dirac equation are

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix}, \tag{3}$$

$$v^{(1)} = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}, \quad v^{(2)} = -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}. \tag{4}$$

(a) [5 points] A convenient normalization for these spinors is $u^\dagger u = \frac{2E}{c}$ (and similarly for v). Show that the resulting normalization factor is the *same* for all the spinors and is given by $N = \sqrt{\frac{(E+mc^2)}{c}}$.

(b) [5 points] Show that $u^{(1)\dagger}u^{(2)} = 0 = v^{(1)\dagger}v^{(2)}$, i.e. $u^{(1)}$ and $u^{(2)}$ are orthogonal, and likewise, $v^{(1)}$ and $v^{(2)}$ are orthogonal. What about $u^{(1)\dagger}v^{(1)}$ and $u^{(2)\dagger}v^{(2)}$?

(c) [5 points] Show that in the non-relativistic limit, the lower two components of $u^{(1)}$ and $u^{(2)}$ are smaller than the upper two components by a factor of $\frac{v}{c}$. Similarly, show that for $v^{(1)}$ and $v^{(2)}$, the upper two components are smaller than the lower two components by a factor of $\frac{v}{c}$.

(d) [5 points] If the z-axis points along the direction of motion, show that $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ are all eigenstates of S_z, where the spin vector is given in Eq. (2).

3. **Helicity Eigenspinors:** [15 points] Starting from the Dirac spinors $u^{(1)}$ and $u^{(2)}$ given in Eq. (3), construct suitable linear combinations $u^{(+)}$ and $u^{(-)}$ so that they are eigenspinors of the helicity operator $(\hat{\boldsymbol{p}} \cdot \boldsymbol{\Sigma})$ with eigenvalues ± 1. Here $\hat{\boldsymbol{p}} \equiv \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$ is the unit vector along the 3-momentum direction and $\boldsymbol{\Sigma}$ is given in Eq. (2).