PHYSICS 474: Introduction to Particle Physics

Homework 5 **Due: 11.30 Monday 02/17/20**

1. $SO(4)$ **Algebra:**

   (a) [5 points] Show that there are $N(N-1)/2$ linearly independent real, antisymmetric $N \times N$ matrices. This means there are $N(N-1)/2$ generators for $SO(N)$.

   [*Hint:* You might find it useful to write the generators as $\mathcal{J}_{(mn)}$ where the m-th row, n-th column is 1; so n-th row, m-th column is -1 to make it anti-symmetric; and the rest are zero. You can think of them as generating rotations in the (mn)-plane. Count how many ways you can do this for an $N \times N$ matrix.]

   (b) [5 points] From part (a), we know there are 6 generators for $SO(4)$, i.e. $\mathcal{J}_{(mn)}$ with $(mn) = \{(12), (23), (31), (14), (24), (34)\}$. Make them Hermitian by inserting an $-i$, i.e. $J_{(mn)} = -i\mathcal{J}_{(mn)}$. Show that their commutators are given by

   $$[J_{(mn)}, J_{(pq)}] \;=\; i\big(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}\big)\,, \tag{1}$$

   where δ_{mn}'s are the usual Kronecker delta functions.

   [*Hint:* It might be easier if you first realize that the elements of the matrix $J_{(mn)}$ can be written as $[J_{(mn)}]_{ij} = -i(\delta_{mi}\delta_{nj} - \delta_{mj}\delta_{ni})$.]

   (c) [5 points] The 6 matrices in part (b) naturally divide into two sets: (i) $\{J_{(23)}, J_{(31)}, J_{(12)}\}$, which generate rotations in the 3-dimensional subspace spanned by the 1-, 2-, and 3-axes, and (ii) $\{J_{(14)}, J_{(24)}, J_{(34)}\}$, which involve rotations involving the 4-th axis. Denote the set (i) as $\{J_1, J_2, J_3\}$ and the set (ii) as $\{K_1, K_2, K_3\}$. Using Eq. (1), show that the $SO(4)$ algebra now becomes

   $$[J_i, J_j] \;=\; i\varepsilon_{ijk}J_k\,, \qquad [J_i, K_j] \;=\; i\varepsilon_{ijk}K_k\,, \qquad [K_i, K_j] \;=\; i\varepsilon_{ijk}J_k\,. \tag{2}$$

2. **LRL Vector and Hydrogen Spectrum:**

   The Lagrange-Runge-Lenz (LRL) vector for a Coulomb-type potential $V(r) = -\frac{\kappa}{r}$ is given by

   $$\mathcal{L} \;=\; \frac{1}{m}(\mathbf{L}\times\mathbf{p}) + \frac{\kappa\mathbf{r}}{r} \;\equiv\; \frac{1}{2m}(\mathbf{L}\times\mathbf{p} - \mathbf{p}\times\mathbf{L}) + \frac{\kappa\mathbf{r}}{r}\,, \tag{3}$$

where $\mathbf{L}$ and $\mathbf{p}$ are the angular momentum and linear momentum, respectively.

(a) [5 points] Show that $\frac{d\mathcal{L}}{dt} = 0$, i.e. $\mathcal{L}$ is a conserved quantity.

(b) [10 points] Show that

$$[L_i, \mathcal{L}_j] = i\hbar\varepsilon_{ijk}\mathcal{L}_k \quad \text{and} \quad [\mathcal{L}_i, \mathcal{L}_j] = -\frac{2H}{m} i\hbar\varepsilon_{ijk}L_k\,, \tag{4}$$

where $H = \frac{p^2}{2m} - \frac{\kappa}{r}$ is the non-relativistic Hamiltonian. [*Hint*: Use $[r_i, p_j] = i\hbar\delta_{ij}$ to get $[L_i, r_j] = i\hbar\varepsilon_{ijk}r_k$ and $[L_i, p_j] = i\hbar\varepsilon_{ijk}p_k$, and use these to derive the desired commutation relations.]

(c) [5 points] Define $M_i \equiv \sqrt{-\frac{m}{2H}}\mathcal{L}_i$. Using your results from part (b), together with the relation $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$ (angular momentum algebra) and Problem 2c, show that L_i and M_i satisfy the $SO(4)$ algebra [as given in Eq. (2) except for an $\hbar$ to match dimensions on both sides].

(d) [5 points] Define $A_{\pm,i} = (L_i \pm M_i)/2$. Show that

$$[A_{+,i}, A_{+,j}] \;=\; i\hbar\varepsilon_{ijk}A_{+,k}\,, \qquad [A_{-,i}, A_{-,j}] \;=\; i\hbar\varepsilon_{ijk}A_{-,k}\,, \qquad [A_{+,i}, A_{-,j}] \;=\; 0\,. \tag{5}$$

(e) [5 points] Show that $\mathbf{A}_+^2 - \mathbf{A}_-^2 = 0$ and $\mathbf{A}_+^2 + \mathbf{A}_-^2 = -\frac{1}{2}\left(\hbar^2 + \frac{m}{2H}\kappa^2\right)$.

(f) [5 points] Replacing the Hamiltonian H with its eigenvalue E and $\mathbf{A}_\pm^2$ with their eigenvalues $a_\pm(a_\pm + 1)$ (with $a_+ = a_- \equiv a$, because $\mathbf{A}_+^2 - \mathbf{A}_-^2 = 0$), show that

$$E \;=\; -\frac{m\kappa^2}{2\hbar^2}\frac{1}{n^2}\,, \tag{6}$$

where $n \equiv 2a + 1 = 1, 2, 3, \cdots$ for $a = 0, \frac{1}{2}, 1, \cdots$.