PHYSICS 474: Introduction to Particle Physics

Homework 3 **Due:** 11.30 Monday 02/03/2020

1. **'Spinning' Electron:** Some popular science books 'visualize' the electron spin by interpreting it literally as a *classical* solid sphere rotating about an axis going through its center.

   (a) [5 points] If the electron has a radius r and is spinning with angular momentum $\hbar/2$, what is the speed of a point on its 'equator'?

   (b) [5 points] Experimentally, we have probed distance scales down to 10^{-18} m and do not find any structure inside the electron. What does this tell us about the speed and this model of 'spinning' electron?

2. **Lie Algebra of $SO(3)$:** The Lie algebra is defined by the commutation relation $[J_i, J_j] = i f_{ijk} J_k$ (sum over k implied), where J_i's are the generators and f_{ijk} are called the *structure constants* of the given Lie group.

   (a) [5 points] Show that if J_i's are Hermitian, then f_{ijk} must be real numbers.

   (b) [5 points] For the $SO(3)$ group, use the explicit 3-dimensional matrix representations of the J_i's discussed in class to show that $f_{ijk} = \varepsilon_{ijk}$ (the Levi-Civita symbol).

   (c) [10 points] Use the orbital angular momentum operator in quantum mechanics: $\boldsymbol{L} = -i\hbar\, \boldsymbol{x} \times \boldsymbol{\nabla}$ to show explicitly that $[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k$, i.e. we can identify the 3 generators of $SO(3)$ with the x, y, z components of the orbital angular momentum.

   (d) [5 points] Given the ladder operators $J_\pm = J_x \pm iJ_y$, use the result from part (b) to find the commutation relations $[J_z, J_\pm]$, $[J_+, J_-]$ and $[J_i, \boldsymbol{J}^2]$.

3. **Clebsch-Gordan Coefficients:**

   (a) [10 points] For an electron in a hydrogen atom with orbital angular momentum quantum number $l = 1$, what are the possible *total* angular momentum quantum number j values? Work out the relevant Clebsch-Gordan coefficients.

   (b) [5 points] For the electron in the $\left|j = \frac{3}{2},\, j_z = \frac{1}{2}\right\rangle$ state, what is the probability of finding it with the z-component of spin $s_z = \frac{1}{2}$?