**PHYSICS 474: Introduction to Particle Physics**

**Homework 9** **Due:** noon Friday, April 6, 2018

1. **Completeness Relation:**

   (a) [5 points] Using the four canonical solutions of the Dirac equation

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix},$$

$$v^{(1)} = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}, \quad v^{(2)} = -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}, \tag{1}$$

   with $N = \sqrt{(E+mc^2)/c}$, prove the completeness relations for Dirac spinors:

$$\sum_{s=1,2} u^{(s)}\bar{u}^{(s)} = \not{p} + mc\,, \qquad \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} = \not{p} - mc\,. \tag{2}$$

   (b) [5 points] Using the polarization vectors

$$\epsilon^{(1)} = (1,0,0)\,, \quad \epsilon^{(2)} = (0,1,0)\,, \tag{3}$$

   and the 3-momentum vector $\mathbf{p} = (0,0,p)$, prove the completeness relation for polarization vectors:

$$\sum_{s=1,2} \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j\,, \tag{4}$$

   where $\hat{\mathbf{p}}$ is the unit vector in the $\mathbf{p}$-direction.

2. **Contractions:** Prove the following contraction identities:

   (a) [1 point] $g_{\mu\nu} g^{\mu\nu} = 4$.

   (b) [2 points] $\not{p}\not{q} + \not{q}\not{p} = 2p \cdot q$.

(c) [2 points] $\gamma_\mu \gamma^\mu = 4$.

(d) [5 points] $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$.

(e) [5 points] $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\mu = 4g^{\nu\lambda}$.

(f) [5 points] $\gamma_\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma^\mu = -2\gamma^\sigma \gamma^\lambda \gamma^\nu$.

3. **Traces:** Prove the following trace identities:

(a) [2 points] $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$.

(b) [5 points] $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$.

(c) [4 points] $\mathrm{Tr}(\gamma^5 \gamma^\mu) = 0$.

(d) [4 points] $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu) = 0$.

(e) [5 points] $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = -4i\varepsilon^{\mu\nu\lambda\sigma}$, where $\varepsilon^{\mu\nu\lambda\sigma}$ is the Levi-Civita tensor in 4-dimensions ($+1$ for even permutations, $-1$ for odd permutations, 0 if any two indices are repeated).