PHYSICS 474: Introduction to Particle Physics

Homework 8 **Due:** noon Friday, March 30, 2018

1. **Dirac Spinors:** The four canonical solutions of the Dirac equation are

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \end{pmatrix},$$
$$v^{(1)} = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{-cp_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix}, \quad v^{(2)} = -N \begin{pmatrix} \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}. \tag{1}$$

(a) [5 points] A convenient normalization for these spinors is $u^\dagger u = \frac{2E}{c}$ (and similarly for v). Show that the resulting normalization factor is the *same* for all the spinors and is given by $N = \sqrt{(E+mc^2)/c}$.

(b) [5 points] Show that $u^{(1)\dagger}u^{(2)} = 0 = v^{(1)\dagger}v^{(2)}$, i.e. $u^{(1)}$ and $u^{(2)}$ are orthogonal, and likewise, $v^{(1)}$ and $v^{(2)}$ are orthogonal. What about $u^{(1)\dagger}v^{(1)}$ and $u^{(2)\dagger}v^{(2)}$?

(c) [5 points] Show that in the non-relativistic limit, the lower two components of $u^{(1)}$ and $u^{(2)}$ are smaller than the upper two components by a factor of $\frac{v}{c}$. Similarly, show that for $v^{(1)}$ and $v^{(2)}$, the upper two components are smaller than the lower two components by a factor of $\frac{v}{c}$.

(d) [5 points] If the z-axis points along the direction of motion, show that $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ are all eigenstates of S_z, where the spin vector is given by

$$\mathbf{S} = \frac{\hbar}{2}\mathbf{\Sigma} \equiv \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{2}$$

2. **Conservation of Total Angular Momentum:** Given the Dirac Hamiltonian

$$H = c\gamma^0(\boldsymbol{\gamma}\cdot\mathbf{p} + mc), \tag{3}$$

show that

(a) [5 points] $[H, \mathbf{L}] = -i\hbar c\gamma^0(\boldsymbol{\gamma} \times \mathbf{p})$, where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the orbital angular momentum.

(b) [5 points] $[H, \mathbf{S}] = i\hbar c\gamma^0(\boldsymbol{\gamma} \times \mathbf{p})$, where $\mathbf{S}$ is the spin angular momentum.

This implies that $[H, \mathbf{J}] = 0$, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum, i.e. $\mathbf{J}$ is a constant of motion.

3. **Lorentz Transformation of Dirac Spinors:**

(a) [10 points] Show that the Lorentz invariance of the Dirac equation requires that under Lorentz transformation, $\psi \to S\psi$ where

$$S = \begin{pmatrix} a_+ & a_-\sigma_1 \\ a_-\sigma_1 & a_+ \end{pmatrix} \tag{4}$$

with $a_\pm = \pm\sqrt{\frac{1}{2}(\gamma \pm 1)}$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

(b) [10 points] Show that the bilinears $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$ are Lorentz-invariant.