Nuclear matter

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Free-fermion model of nuclear matter

Nuclear matter consists of protons, neutrons, electrons

<table>
<thead>
<tr>
<th>particle</th>
<th>mass</th>
<th>charge</th>
<th>nucleon number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$m_p = 938\ \text{MeV}$</td>
<td>$+1$</td>
<td>1</td>
</tr>
<tr>
<td>$n$</td>
<td>$m_n = 939\ \text{MeV}$</td>
<td>$0$</td>
<td>1</td>
</tr>
<tr>
<td>$e^-$</td>
<td>$m_e = 0.5\ \text{MeV}$</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>
Three independent species

Simple non-interacting model: each species fills its Fermi sphere up to its Fermi momentum

\[ E_{F,p} = \mu_p = \sqrt{p_{Fp}^2 + m_p^2} \approx m_p + \frac{p_{Fp}^2}{2m_p} \]
\[ E_{F,n} = \mu_n = \sqrt{p_{Fn}^2 + m_n^2} \approx m_n + \frac{p_{Fn}^2}{2m_n} \]
\[ E_{F,e} = \mu_e = \sqrt{p_{Fe}^2 + m_e^2} \]

Since at typical nuclear densities \( p_{Fn} \ll m_n \) and \( p_{Fp} \ll m_p \), we can treat protons and neutrons as nonrelativistic fermions.

Pressure of a degenerate gas of protons, neutrons, electrons is a function of three chemical potentials

\[ P_{npe}(\mu_p, \mu_n, \mu_e) = P_{NR}(\mu_p, m_p) + P_{NR}(\mu_n, m_n) + P(\mu_e, m_e) \]
Beta equilibrated nuclear matter

Weak interactions allow

\[ n \rightarrow p \ e^- \ \bar{\nu}_e \quad p \ e^- \rightarrow n \ \nu_e \]

and the neutrinos escape, so weak interactions take us to “beta equilibrium”

\[ \mu_n = \mu_p + \mu_e \]

So, pressure of \textit{beta-equilibrated} nuclear matter is a function of \textbf{two} chemical potentials

\[ P_\beta(\mu_n, \mu_e) = P_{NR}(\mu_n - \mu_e, m_p) + P_{NR}(\mu_n, m_n) + P(\mu_e, m_e) \]

Electric charge density in units of e

\[ Q_\beta(\mu_n, \mu_e) = - \frac{\partial P_\beta}{\partial \mu_e} \bigg|_{\mu_n} \]

Nucleon density

\[ N_\beta(\mu_n, \mu_e) = \frac{\partial P_\beta}{\partial \mu_n} \bigg|_{\mu_e} \]
Infinite nuclear matter: electrical neutrality

Uniform infinite matter must be electrically neutral. i.e., charge density \( Q_\beta = 0 \). This fixes \( \mu_e \) to be \( \mu_{e,\text{neutral}}(\mu_n) \) where

\[
Q_\beta(\mu_n, \mu_{e,\text{neutral}}(\mu_n)) = 0
\]

So the pressure of neutral, beta-equilibrated nuclear matter is a function of one chemical potential

\[
P_{\beta,\text{neutral}}(\mu_n) = P_\beta(\mu_n, \mu_{e,\text{neutral}}(\mu_n))
\]

with nucleon density

\[
N_{\beta,\text{neutral}}(\mu_n) = N_\beta(\mu_n, \mu_{e,\text{neutral}}(\mu_n)) = \frac{dP_{\beta,\text{neutral}}}{d\mu_n}
\]
Nuclei: Thomas-Fermi approx

Uniform infinite matter is electrically neutral, but finite blobs of nuclear matter (e.g. nuclei) can be charged. Can we estimate their charge distribution?

μ_e determines the local charge density, but it is also determined by the overall charge distribution, because each electron has charge −e, so

\[ \mu_e = eV \quad (V = \text{electrostatic potential}) \]

\[ \nabla^2 V(x) = -eQ(x) \quad (\text{since } \nabla \cdot \vec{E} = eQ, \quad \vec{E} = -\nabla V) \]

\[ \Rightarrow \nabla^2 \mu_e(x) = -4\pi\alpha Q_\beta(\mu_n(x), \mu_e(x)) \]

\[ \alpha = e^2 / 4\pi \approx 1/137 \] is the electromagnetic coupling.

For a given background nucleon distribution \( \mu_n(x) \), you can solve this diffeq to get \( \mu_e(x) \) and hence the self-consistent charge distribution.
Spherical nucleus, Thomas-Fermi approx

Assume:

- spherical nucleus of radius \( R \)
- nucleus has uniform density so \( \mu_n \approx \begin{cases} \mu_{n,NM} & r < R \\ 0 & r > R \end{cases} \)
- nucleus has total charge \( Z \) so for \( r > R \), \( \mu_e(r) = \frac{Z \alpha}{r} \)

To find \( \mu_e(r) \) inside the nucleus we solve

\[
\nabla^2 \mu_e(r) = -4\pi \alpha Q_\beta(\mu_{n,NM}, \mu_e(r))
\]

with boundary conditions

\[
\frac{d\mu_e}{dr}(r=0) = 0 \quad \text{no elec field at origin}
\]

\[
\mu_e(R) = \frac{Z \alpha}{R} \quad \text{match to external Coulomb potential}
\]
Thomas-Fermi: solving Poisson eqn

“Shooting” method: vary $\mu_e(0)$ until we get a solution of

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\mu_e}{dr} \right) = -4\pi\alpha Q_\beta(\mu_e(r)) \quad \frac{d\mu_e}{dr} \bigg|_{r=0} = 0$$

that matches to $\mu_e(R) = \frac{Z\alpha}{R}$. 