Basic statistical mechanics of fermions

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Zero temperature, one fermion species

What do the following have in common:

- a box of ^{3}He atoms
- a box of electrons (e.g. valence electrons in a lump of metal)
- a box of neutrons (e.g. a piece of the crust of a neutron star)
- a box of quarks (perhaps part of the core of a neutron star?)

Apart from energy, they all have an additional conserved quantity: the number of fermions.

The thermodynamic description of such a system involves 4 quantities, but they are related to each other to form a 1-parameter family:

energy density	ε	
fermion density	n	(e.g. electron density)
chemical potential	μ	
pressure	P	

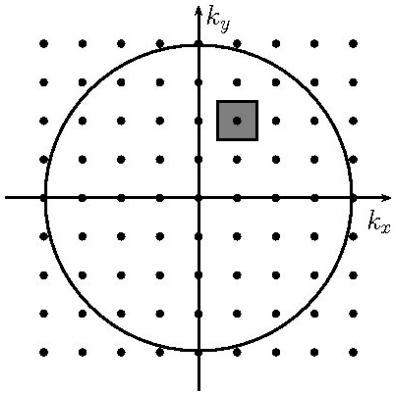
The Fermi momentum

We will use $\hbar = c = 1$ units.

For fermions in a box of volume V, there is one momentum state in every $\dfrac{(2\pi)^3}{V}$ of momentum space. (This follows from the quantum mechanics of particles in a box.)

Since fermions have spin $\frac{1}{2}$, we can put 2 particles in every momentum state ($S_z = \pm\frac{1}{2}$).

At zero temperature, fermions fill all the the lowest energy states up to the Fermi momentum p_F (the "Fermi sphere").



At $T = 0$ the Fermi momentum completely specifies the state of the system. All thermodynamic properties (n, ε, μ, P) are functions of p_F.

Particle number density

The number of fermions in a box of volume V is

$$N = \underbrace{2}_{\text{spin states}} \times \frac{\text{momentum volume of filled states}}{\text{momentum volume of each state}}$$

i.e.,

$$N = 2 \times \frac{V}{8\pi^3} \int_0^{p_F} d^3p = \frac{2V}{8\pi^3} \frac{4}{3}\pi {p_F}^3$$

So the density of particles is

$$n(p_F) \equiv \frac{N}{V} = \frac{{p_F}^3}{3\pi^2} \tag{1}$$

Energy density

To calculate the total energy of the particles we add up the energies of all the particles in the states within the Fermi sphere,

$$E = 2 \times \frac{V}{8\pi^3} \int_0^{p_F} e(p)\, d^3p$$

where each single-particle state of momentum p has energy $e(p)$.

For an isotropic system there is spherical symmetry, so $d^3p = 4\pi p^2 dp$. So the energy density of the fermions in the box is

$$\varepsilon(p_F) \equiv \frac{E}{V} = \frac{1}{\pi^2} \int_0^{p_F} e(p)\, p^2 dp \tag{2}$$

For free relativistic fermions, $e(p) = \sqrt{p^2 + m^2}$.

Chemical potential

The chemical potential μ is

- the energy of the last particle that was added to the box (when filling up the energy states from lowest to highest). I.e.,

$$\mu = \frac{dE}{dN} = \frac{d\varepsilon}{dn} \tag{3}$$

- the energy of a particle with momentum p_F, i.e. the Fermi energy

$$\mu = e(p_F)$$

Translating between μ and p_F

Up to now we've expressed things like number density and energy density as functions of the Fermi momentum. If we want to express them as functions of the chemical potential then we need to know p_F as a function of μ.

For free particles this is easy, since μ is the Fermi energy, so for a free relativistic particle of mass m,

$$\begin{aligned} \mu = e(p_F) &= \sqrt{{p_F}^2 + m^2} \\ \Rightarrow p_F(\mu) &= \sqrt{\mu^2 - m^2} \end{aligned} \tag{4}$$

Pressure

The pressure P is the force per unit area, $F/A = dE/dV$, at *fixed particle number* (particles can't go through the walls of the box).

- $$P = -\left.\frac{dE}{dV}\right|_N$$
- $$P = \mu n - \varepsilon \tag{5}$$

Usually the pressure is expressed as a function of the chemical potential. Since we previously expressed n and ε in terms of p_F, we would write the pressure as

$$P(\mu) = \mu\, n(p_F(\mu)) - \varepsilon(p_F(\mu))$$

Exercise: Using what we have covered in these slides, show that

$$\frac{dP}{d\mu} = n \tag{6}$$