10/8/10

QUANTUM MECHANICS I (523)
PROBLEM SET 6 (hand in October 22)

21) Consider a free particle in one dimension in part a) and a particle with a Hamiltonian $H = \boldsymbol{p}^2/2m + V(\boldsymbol{x})$ in part b).

a) For the case of the one-dimensional problem, consider the position operator in the Heisenberg picture $x_H(t)$. Evaluate

$$[x_H(t), x_H(t=0)] \, .$$

b) Now working in three dimensions, calculate

$$[\boldsymbol{x}_H \cdot \boldsymbol{p}_H, H_H]$$

to obtain

$$\frac{d}{dt}\langle \boldsymbol{x} \cdot \boldsymbol{p} \rangle = \left\langle \frac{\boldsymbol{p}^2}{m} \right\rangle - \langle \boldsymbol{x} \cdot \nabla V \rangle .$$

In order to identify this results as the quantum analog of the virial theorem, the left-hand side should vanish. Under what condition does this happen?

22) A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, the state is represented by the position eigenket $|R\rangle$ ($|L\rangle$), where spatial variations within each half of the box are neglected. The most general state ket can then be written as

$$|\psi\rangle = |R\rangle \langle R|\psi\rangle + |L\rangle \langle L|\psi\rangle,$$

where $\langle R|\psi\rangle$ and $\langle L|\psi\rangle$ can be regarded as "wave functions." The particle can tunnel through the partition; this tunneling effect is characterized by a Hamiltonian

$$H = \Delta \left(|L\rangle \langle R| + |R\rangle \langle L| \right),$$

where Δ is real with dimension of energy.

a) Determine the normalized energy eigenkets and eigenvalues.

b) In the Schrödinger picture the basis kets $|R\rangle$ and $|L\rangle$ are fixed, and the state ket evolves in time. Suppose at $t = 0$ the system is represented by $|\psi(t = 0)\rangle$ as given above. Determine the state ket $|\psi(t)\rangle$ by using the results of a).

c) Suppose at $t = 0$ the particle is on the right side with certainty. What is the probability for observing the particle on the left side as a function of time?

d) Suppose by mistake the Hamiltonian was written as

$$H = \Delta |L\rangle \langle R| .$$

Solve the general time-evolution problem with this Hamiltonian and demonstrate that probability is not conserved.

23) Consider the one-dimensional harmonic oscillator. Do the following without using wave functions.

a) Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle q\rangle$ is as large as possible.

b) Assume that at $t = 0$ the system is in this state. Determine the time-evolved state at t in the Schrödinger picture and evaluate the expectation value $\langle q\rangle$ as a function of time in both the Schrödinger **and** Heisenberg picture.

c) Evaluate $\langle (\Delta q)^2\rangle$ using either picture.

24) Consider a particle with mass m in a one-dimensional potential of the following form:

$$V = \begin{cases} \frac{1}{2}kq^2 & \text{for } q > 0 \\ \infty & \text{for } q < 0. \end{cases}$$

a) Determine the ground state energy by "thinking outside the box."

b) Determine the expectation value $\langle q^2\rangle$ for the ground state.