09/26/14

QUANTUM MECHANICS I (523)
PROBLEM SET 4 (hand in October 3)

14) (10 points) Construct the transformation matrix that connects the basis in which S_z is diagonal to the one in which S_x is diagonal. Demonstrate that your result is consistent with the general relation

$$U = \sum_i |b_i\rangle \langle a_i|$$

which was discussed in class.

15) (practice problem: don't hand in)

a) Consider the classical position and linear momentum in the x-direction denoted by x and p_x, respectively. Evaluate the classical Poisson bracket

$$[x, F(p_x)]_{PB} \,,$$

with F a function of the momentum.

b) Consider the position operator x_i representing the x, y, or z operator and a well-behaved function $G(\boldsymbol{p})$ of the momentum operator (in three dimensions). Derive

$$[x_i, G(\boldsymbol{p})] = i\hbar \frac{\partial G}{\partial p_i},$$

by making a suitable expansion of the function G.

c) Evaluate

$$\left[x^2, p_x^2\right]$$

and compare with the corresponding classical Poisson bracket

$$\left[x^2, p_x^2\right]_{PB} .$$

16) (10 points) Consider the operator x in the momentum representation (in one dimension). Prove

$$\langle p'|\, x\, |\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle,$$

and

$$\langle\beta|\, x \,|\alpha\rangle = i\hbar \int dp'\ \phi^*_\beta(p') \frac{\partial}{\partial p'} \phi_\alpha(p'),$$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum space wave functions.

17) (20 points) Consider a particle in the ground state of an infinite well of width a (you should have seen this problem and its solution before),

$$\begin{aligned} \psi(x) &= \sqrt{\tfrac{2}{a}}\cos(\pi x/a) \quad (-a/2 < x < a/2) \\ &= 0 \quad\quad\quad\quad\quad\quad\quad (x < -a/2,\ x > a/2) \end{aligned}$$

At time $t = 0$ we suddenly take away the "walls" of the well, setting $V(x) = 0$ everywhere. The state is still $\psi(x)$, but it is now appropriate to write this state as a superposition of plane waves,

$$\psi(x) = \int_{-\infty}^{\infty} \tilde{\psi}(k) \frac{\exp(ikx)}{\sqrt{2\pi}}\, dk\ . \qquad (*)$$

a) Calculate $\tilde{\psi}(k)$, the Fourier transform of $\psi(x)$. Calculate $|\tilde{\psi}(k)|^2$. Check that $|\tilde{\psi}(k)|^2$ is normalized correctly: the easiest way is to evaluate the integral numerically (using MATLAB, Mathematica, other software, or your own program) for various values of a. Make a plot of $|\tilde{\psi}(k)|^2$ for k from 0 to 0.2, for $a = 100$.

b) Calculate k_0, the lowest value of k at which $|\tilde{\psi}(k)|^2 = 0$. As the infinite well becomes wider (increasing a), what happens to k_0? Note from your plot of $|\tilde{\psi}(k)|^2$ that the dominant contribution to $\tilde{\psi}$ comes from $0 < k < k_0$, so k_0 is an estimate of Δk, the range of wavenumbers of the plane waves that constitute $\psi(x)$. The uncertainty in position is $\Delta x = a$. Write down $\Delta x \Delta k$ and hence $\Delta x \Delta p$. Is your result consistent with Heisenberg's uncertainty relation?