09/12/14

QUANTUM MECHANICS I (523)
PROBLEM SET 2 (hand in September 19)

6) (10 points) Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$$

and

$$\{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

where

- $S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$
- $S_y = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)$
- $S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$ .

7) (10 points) Construct kets $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle$ such that

$$\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\,|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle = \frac{\hbar}{2}\,|\boldsymbol{S}\cdot\hat{\boldsymbol{n}};+\rangle\,,$$

where $\hat{\boldsymbol{n}}$ is characterized by two angles $\alpha$ and $\beta$. $\alpha$ corresponds to the azimuthal angle (measured from the $x$-axis) and $\beta$ is the polar angle. Express your answer as a linear combination of $|+\rangle$ and $|-\rangle$ kets. Make sure you solve the actual eigenvalue problem!

8) (10 points) Consider a beam with spin 1 atoms yielding three possible outcomes for a Stern-Gerlach magnet in the $\hat{z}$-direction. The corresponding eigenvalue equations for the $\hat{z}$-component of the spin are then

$$S_z\,|1\rangle = \hbar\,|1\rangle$$
$$S_z\,|0\rangle = 0\hbar\,|0\rangle$$
$$S_z\,|-1\rangle = -\hbar\,|-1\rangle\,.$$

a) Construct the matrices representing the eigenkets in this basis as well as the matrix representation of $S_z$.

b) From experiments on such a system one may infer that the $S_x$ eigenstates can be written as

$$|1_x\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$
$$|0_x\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$$
$$|-1_x\rangle = \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle .$$

Check that these states are orthogonal and construct their matrix representations in the $z$-basis. Also construct the matrix representing the operator $S_x$ in the $z$-basis.

c) A spin-1 system is prepared in the state

$$|\psi\rangle = \frac{2}{\sqrt{6}}|1\rangle - \frac{i}{\sqrt{6}}|0\rangle + \frac{i}{\sqrt{6}}|-1\rangle .$$

Find the probabilities of measuring each of the possible spin components in the $z$-direction.

9) (10 points) A beam of spin 1/2 atoms goes through a series of Stern-Gerlach-type measurements as follows:

1) The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms, where $s_z$ denotes the eigenvalue of the operator $S_z$.
2) The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where $s_n$ is the eigenvalue of the operator $\boldsymbol{S} \cdot \hat{\boldsymbol{n}}$, with $\hat{\boldsymbol{n}}$ making an angle $\beta$ in the $xz$-plane with respect to the $z$-axis.
3) The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must one orient the second measuring apparatus in order to maximize the intensity of the final $s_z = -\hbar/2$ beam?