3/2/18

MECHANICS (411)
PROBLEM SET 7 (hand in March 23 at the beginning of class)

23) (10 points)

a) Write down the Lagrangian, $L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ for two particles of equal mass, $m_1 = m_2 = m$, confined to the x axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$. Here x is the extension of the spring, $x = x_1 - x_2 - \ell$, where ℓ is the spring's natural length.

b) Rewrite the Lagrangian in terms of the center of mass position $X = \frac{x_1+x_2}{2}$, and the extension x. Write down the two Lagrange equations for X and x and solve for $X(t)$ and $x(t)$. Describe the motion.

24) (10 points) Consider two particles interacting by a Hooke's law potential energy, $U = \frac{1}{2}kr^2$, where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ is their relative position, and subject to no external forces. Show that $\boldsymbol{r}(t)$ describes an ellipse. Hence, show that both particles move on similar ellipses around their common CM. This is the other instance of an interaction admitting closed orbits.

Hint: Perhaps the easiest way to solve the problem is to use cartesian coordinates, x, y, in the plane of the orbit. Your solution will look like $x = a\cos\omega t + b\sin\omega t$, and a similar expression for y. You can solve for $\cos\omega t$ and $\sin\omega t$, and put the equation of the orbit in the form $\alpha x^2 + 2\beta xy + \gamma y^2 = \kappa$, which defines an ellipse if $\alpha > 0$, $\gamma > 0$ and $\alpha\gamma > \beta^2$.

25) (10 points) Imagine that the Earth were moving on a circular orbit around the Sun. Show that if the mass of the Sun decreases abruptly to half its value, the Earth's orbit becomes parabolic. You may assume that the reduced mass remains equal to m_E the mass of Earth.

26) (10 points) Suppose that we decide to send a spacecraft to Neptune, using the simple transfer described in Example 8.6 (page 318 in the textbook). The craft starts in a circular orbit close to the earth (radius 1 AU or astronomical unit) and is to end up in a circular orbit near Neptune (radius about 30 AU). Use Kepler's third law to show that the transfer will take about 31 years. (In practice we can do a lot better than this by arranging that the craft gets a gravitational boost as it passes Jupiter).