MECHANICS (411)
PROBLEM SET 3 (hand in February 9 at the beginning of class)

9) (5 points) A meteorite falls to earth from very far away with negligible initial speed. Neglecting air friction, how fast is it going when it hits the earth's surface? Express your answer in terms of the mass of the earth M_E, the radius of the earth R_E, and the gravitational constant G. You may take the initial distance from the earth to be infinite.

10) (10 points) Suppose a particle of mass m moves along the x-axis governed by a force $F(x) = -ax + bx^3$, where a and b are positive constants.

   a) Find the potential $U(x)$ and draw a rough graph of it.
   b) What is the minimum speed the particle must be given at $x = 0$ so that it will escape to $x = +\infty$?

11) (10 points) A very useful application of the gradient is that it gives us the change in U (or any scalar function) resulting from a small displacement $\mathrm{d}\boldsymbol{r}$:

$$\mathrm{d}U = \boldsymbol{\nabla} U \cdot \mathrm{d}\boldsymbol{r}. \qquad (1)$$

   a) Show that the direction of $\boldsymbol{\nabla} U$ at any point $\boldsymbol{r}$ is the direction in which U increases fastest as we move away from $\boldsymbol{r}$. (Choose a small displacement $\mathrm{d}\boldsymbol{r} = \epsilon \boldsymbol{u}$, where $\boldsymbol{u}$ is a unit vector and ϵ is fixed and small, and find the direction of $\boldsymbol{u}$ for which $\mathrm{d}U$ is largest).
   b) Which of the following forces is conservative? (i) $\boldsymbol{F} = k(x, 2y, 3z)$ where k is a constant. (ii) $\boldsymbol{F} = k(y, x, 0)$. (iii) $\boldsymbol{F} = k(-y, x, 0)$. For those which are conservative, find the corresponding potential energy U, and verify by direct differentiation that $\boldsymbol{F} = -\boldsymbol{\nabla} U$.

12) (15 points) Consider a mass m confined to the x-axis with a potential energy of $U = kx^4$ with $k > 0$. At time $t = 0$, when it is sitting at the origin, it is given a sudden kick to the right.

   a) Find the time for the mass to reach its maximum displacement $x_{max} = A$. Your answer will be given as an integral over x in terms of m, A and k.
   b) Find the period τ of oscillations of amplitude A, and by making a suitable change of variables, show that it is inversely proportional to A. (Thus, the larger the amplitude the shorter the period!)

c) The integral cannot be evaluated in terms of elementary functions. This is often the case, but for small oscillations about the minimum of any potential energy $U(x)$, we may approximate U by the first three terms of its Taylor series in powers of x (or if the minimum was located at $x = a$, in powers of $x - a$). Write down the equation of conservation of energy. The small oscillations about the minimum are thus approximately simple harmonic. If

$$U = \frac{x(x-3)^2}{3}, \tag{2}$$

find the period of small oscillations about the minimum at $x = 3$. (Note that there is another equilibrium point at $x = 1$, but it is a maximum.)