Physics 217

Homework 5

1. For the infinite square well the energies are given by $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$. $\nu = \frac{\Delta E}{h} = \frac{E_n - E_1}{h} = (n^2 - 1)\frac{\pi\hbar}{4ma^2}$. Now use $\lambda\nu = c$ and get $\lambda = \frac{4mca^2}{\pi\hbar(n^2-1)}$.

2. $\psi_n(x) = \sqrt{\frac{2}{a}}\cos(\frac{n\pi x}{a})$ for odd n and $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a})$ for even n. There are three cases we must address in this problem:

   (a) n and m both odd: $\int_{-a/2}^{a/2}\psi_n^\star(x)\psi_m(x)dx = 2\int_0^{a/2}\psi_n^\star\psi_m dx$. Making the substitution $z = \frac{\pi x}{a}$ and integrating we get an answer of $\frac{4}{\pi}\left[\frac{m\sin(n\pi/2)\cos(m\pi/2) - n\cos(n\pi/2)\sin(m\pi/2)}{n^2 - m^2}\right]$. Since both n and m are odd all of the cosine terms in the numerator give zero and this case is proved.

   (b) n and m both even: Using the same method as above you get a result of $\int_{-a/2}^{a/2}\psi_n^\star(x)\psi_m(x)dx = \frac{4}{\pi}\left[\frac{n\sin(n\pi/2)\cos(m\pi/2) - m\cos(n\pi/2)\sin(m\pi/2)}{n^2 - m^2}\right]$. This time since both n and m are even all of the sine terms in the numerator vanish and this case is proved.

   (c) This is actually two cases which are the same, either n is odd and m is even or vise versa. In either case you have an odd function being integrated over even limits which is identically zero, and this case is proved.

   Since we have proven this for all possible cases we have proven that this identity is always true.

3. (a) $\int_{-\infty}^\infty |\psi(x)|^2 = 1 = \int_{-\infty}^\infty (A\psi_1^\star + B\psi_2^\star)(A\psi_1 + B\psi_2)dx$. Because the eigenfunctions of the infinite square well are orthogonal the cross terms do not contribute. Since ψ_1 and ψ_2 are normalized eigenfunctions of the infinite square well $\int_{-\infty}^\infty \psi_2^\star\psi_2 dx = \int_{-\infty}^\infty \psi_1^\star\psi_1 = 1$. Thus $\int_{-\infty}^\infty |\psi(x)|^2 = 1 = |A|^2 + |B|^2$.

   (b) $\langle \hat{H} \rangle = \langle \frac{p^2}{2m} + V(x) \rangle = -\frac{\hbar^2}{2m}\langle \frac{\partial^2}{\partial x^2} \rangle$ for a particle inside of the infinite square

well. $\langle \hat{H} \rangle = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} (A\psi_1^\star + B\psi_2^\star)\frac{\partial^2}{\partial x^2}(A\psi_1 + B\psi_2)dx$. After some algebra you arrive at $\langle \hat{H} \rangle = \int_{-\infty}^{\infty}\left[\frac{(1)^2\pi^2\hbar^2}{2ma^2}|A|^2\psi_1^\star\psi_1 + \frac{(2)^2\pi^2\hbar^2}{2ma^2}|B|^2\psi_2^\star\psi_2\right]dx$, which gives $\langle \hat{H} \rangle = \frac{(1)^2\pi^2\hbar^2}{2ma^2}|A|^2 + \frac{(2)^2\pi^2\hbar^2}{2ma^2}|B|^2$.

According to the measurement postulates a measurement of energy can yield only energies which are eigenvalues of the Hamiltonian. Thus you can see that you would measure energy E_1 with a probability of $|A|^2$ and E_2 with a probability of $|B|^2$ and the expectation value of the energy matches the previous result $\langle H \rangle = |A|^2 E_1 + |B|^2 E_2$.

4. $\psi_2(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)$ and $\psi_3(\mathrm{x}) = \sqrt{\frac{2}{a}}\cos\left(\frac{3\pi \mathrm{x}}{\mathrm{a}}\right)$. The probability of the electron being in the interval $[-a/2, 0]$ is given by $P = \int_{-a/2}^{0}|\Psi|^2 dx$. The multiplication of $\Psi^*\Psi$ will give four terms. The time dependence of the "cross" terms will not cancel because the energy values are different. The two terms without time dependence give results of $\int_{-a/2}^{0}\cos^2\left(\frac{3\pi x}{a}\right)dx = \frac{1}{2}\int_{-a/2}^{a/2}\cos^2\left(\frac{3\pi x}{a}\right)dx = \frac{1}{2}\int_{-a/2}^{a/2}\sin^2\left(\frac{2\pi x}{a}\right)dx = \frac{a}{4}$. Next one must deal with the time dependent terms. These terms have common x dependent functions thus $\int_{-a/2}^{0}[\psi_2^*\psi_3 e^{i(E_2-E_3)t/\hbar} + \psi_3^*\psi_2 e^{i(E_3-E-2)t/\hbar}]dx = 2\cos\left[\frac{E_2-E_3}{\hbar}t\right]\int_{-a/2}^{0}\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{3\pi x}{a}\right)dx = -.6$ where I have made use of the identity $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$. Putting all parts together you get $Prob. = \frac{1}{2} - \frac{1.2a}{\pi}\cos(\omega t)$ where $\omega = \frac{E_2-E_3}{\hbar}$. Cosine is a function which oscillates with a frequency of ω thus the period of oscillation is $T = 2\pi/\omega = \frac{h}{E_2-E_3}$.