Symmetric and antisymmetric states

When is quantum physics expected?

Consider the energy levels for a particle of mass m enclosed in a box with volume $V = L^3$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$$ positive integers

Total number of states below energy E

$$\Omega(E) = \frac{\pi}{6}\left(\frac{8mL^2E}{h^2}\right)^{3/2} = \frac{\pi}{6}\left(\frac{8mE}{h^2}\right)^{3/2} V$$

"Quantumness" --> indistinguishability not important when

$$1 \gg Q \equiv \frac{N}{\Omega} = \frac{6}{\pi}\rho\left(\frac{h^2}{12mk_BT}\right)^{3/2}$$

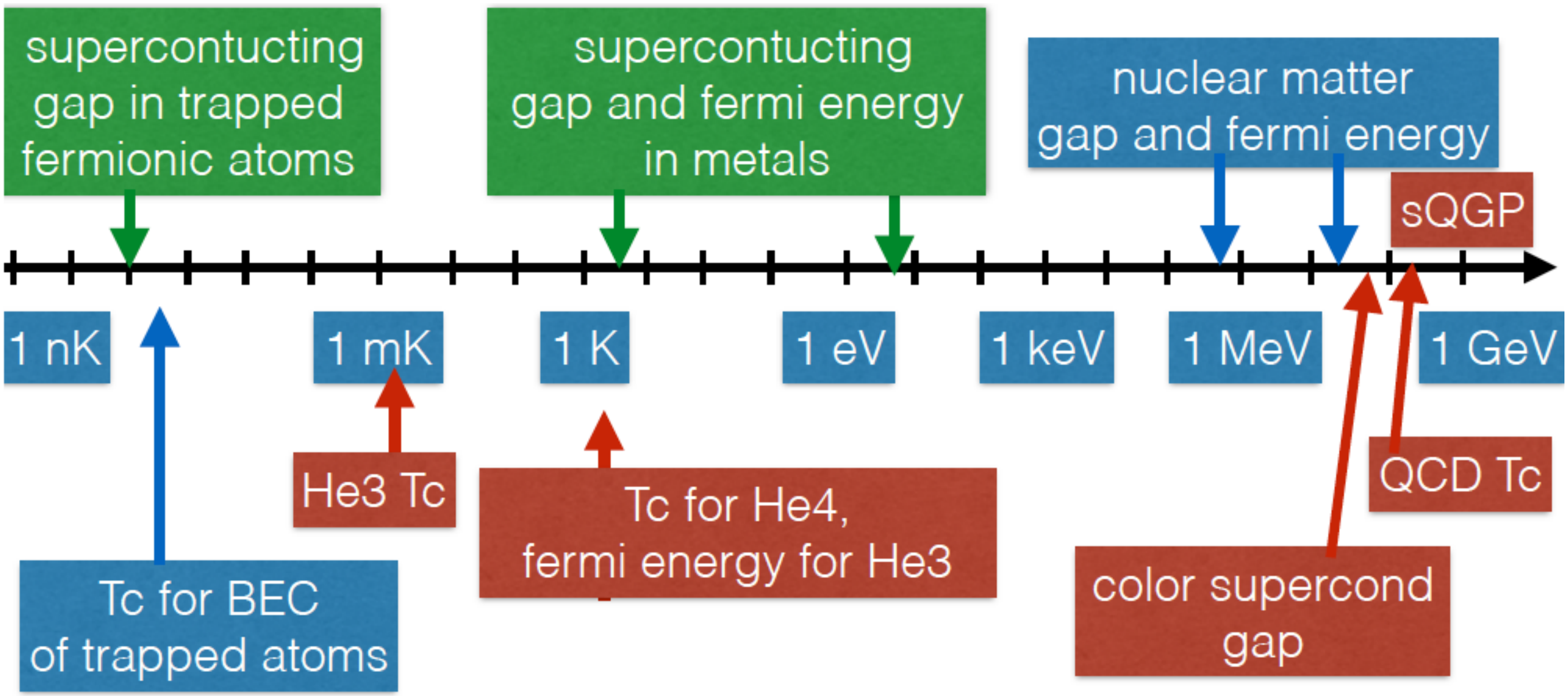
Use $E = \frac{3}{2}k_BT$

Q

System	T (K)	Density (m^{-3})	Mass (u)	Q
He (l)	4.2	1.9×10^{28}	4.0	1.1
He (g)	4.2	2.5×10^{27}	4.0	1.4×10^{-1}
He (g)	273	2.7×10^{25}	4.0	2.9×10^{-6}
Ne (l)	27.1	3.6×10^{28}	20.2	1.1×10^{-2}
Ne (g)	273	2.7×10^{25}	20.2	2.5×10^{-7}
e^- Na metal	273	2.5×10^{28}	5.5×10^{-4}	1.7×10^{3}
e^- Al metal	273	1.8×10^{29}	5.5×10^{-4}	1.2×10^{4}
e^- white dwarfs	10^{7}	10^{36}	5.5×10^{-4}	8.5×10^{3}
p,n nuclear matter	10^{10}	1.7×10^{44}	1.0	6.5×10^{2}
n neutron star	10^{8}	4.0×10^{44}	1.0	1.5×10^{6}
^{87}Rb condensate	10^{-7}	10^{19}	87	1.5

A wider temperature perspective

- From Shuryak "Quantum Many-Body Physics in a Nutshell"



- 1 eV —> 1.160 452 21(67) x 10^4K (NIST value)

Bosons and Fermions

- Use experimental observations to conclude about consequences of identical particles
- Two possibilities
  - antisymmetric states $\Rightarrow$ fermions half-integer spin
    - Pauli from properties of electrons in atoms
  - symmetric states $\Rightarrow$ bosons integer spin
    - Considerations related to electromagnetic radiation (photons)
- Can also consider quantization of "field" equations
  - e.g. quantize "free" Maxwell equations (see Phys 524)

Wolfgang Pauli (1900-1958)

- The Nobel Prize in Physics 1945 was awarded to Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle".

- paper Zeitschr. f. Phys. 31, 765 (1925)

Review single-particle states

- Notation $|\ldots\rangle$
- ... list of quantum numbers associated with a CSCO
- Normalization $\langle\alpha|\beta\rangle = \delta_{\alpha,\beta}$
- Continuous quantum numbers
  - Example $\langle \boldsymbol{r}, m_s | \boldsymbol{r}', m_s' \rangle = \delta(\boldsymbol{r} - \boldsymbol{r}')\delta_{m_s,m_s'}$
- Completeness $\sum_{\alpha} |\alpha\rangle\langle\alpha| = 1$

Consequences for two-particle states

- CVS for two particles: product space
- Notation $|\alpha_1\alpha_2) = |\alpha_1\rangle|\alpha_2\rangle$
- Orthogonality $(\alpha_1\alpha_2|\alpha_1'\alpha_2') = \delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}$
- Completeness $\sum_{\alpha_1\alpha_2} |\alpha_1\alpha_2)(\alpha_1\alpha_2| = 1$

Exchange degeneracy

- Consider $\alpha_1 \neq \alpha_2$
- Then $|\alpha_2\alpha_1) \neq |\alpha_1\alpha_2)$
- All states $|\alpha_1\alpha_2)$

$$|\alpha_2\alpha_1)$$

$$c_1|\alpha_1\alpha_2) + c_2|\alpha_2\alpha_1)$$

yield α_1 for one particle and α_2 for the other upon measurement

- Yet, unclear which state describes this system and therefore inconsistent with quantum postulates
- Consider permutation operator

$$P_{12}|\alpha_1\alpha_2) = |\alpha_2\alpha_1)$$

with $P_{12} = P_{21} \text{ and } P_{12}^2 = 1$

- Hamiltonian for two particles is symmetric for $1 \Leftrightarrow 2$

Development

- Typical Hamiltonian $H = \frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} + V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$
- Consider operator acting on particle 1 and corresponding eigenvalue $A_1|\alpha_1\alpha_2) = a_1|\alpha_1\alpha_2)$
- Similarly, the same operator acting on particle 2 yields
$$A_2|\alpha_1\alpha_2) = a_2|\alpha_1\alpha_2)$$
- Note $P_{12}A_1|\alpha_1\alpha_2) = a_1 P_{12}|\alpha_1\alpha_2) = a_1|\alpha_2\alpha_1) = A_2|\alpha_2\alpha_1)$
- and $P_{12}A_1|\alpha_1\alpha_2) = P_{12}A_1P_{12}^{-1}P_{12}|\alpha_1\alpha_2) = P_{12}A_1P_{12}^{-1}|\alpha_2\alpha_1)$
- Holds for any state; therefore $P_{12}A_1P_{12}^{-1} = A_2$
- It follows that $P_{12}HP_{12}^{-1} = H$ or $[P_{12}, H] = 0$

Symmetric and antisymmetric two-particle states

- So $$[P_{12}, H] = 0$$
- Common eigenkets either

$$|\alpha_1\alpha_2\rangle_+ = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$$

or

$$|\alpha_1\alpha_2\rangle_- = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) - |\alpha_2\alpha_1)\}$$

- Eigenstates of the Hamiltonian either symmetric $\Rightarrow$ bosons

  or antisymmetric $\Rightarrow$ fermions

- Two-boson state

$$|\alpha_1\alpha_2\rangle_S = \left[\frac{1}{2n_\alpha!n_{\alpha'}!\ldots}\right]^{1/2}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$$

$$\alpha_1 = \alpha_2 = \alpha \Rightarrow |n_\alpha = 2\rangle = |\alpha\alpha\rangle_S = |\alpha\rangle\,|\alpha\rangle$$

$$\alpha_1 \neq \alpha_2 \Rightarrow |\alpha_1\alpha_2\rangle_S = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$$

Fermions

- Antisymmetry: $|\alpha_2\alpha_1\rangle = -|\alpha_1\alpha_2\rangle$
- Both kets represent the same physical state: count only once in completeness relation ⇒ "order" quantum numbers

$$|1\rangle\,,|2\rangle\,,|3\rangle\,,\ldots$$

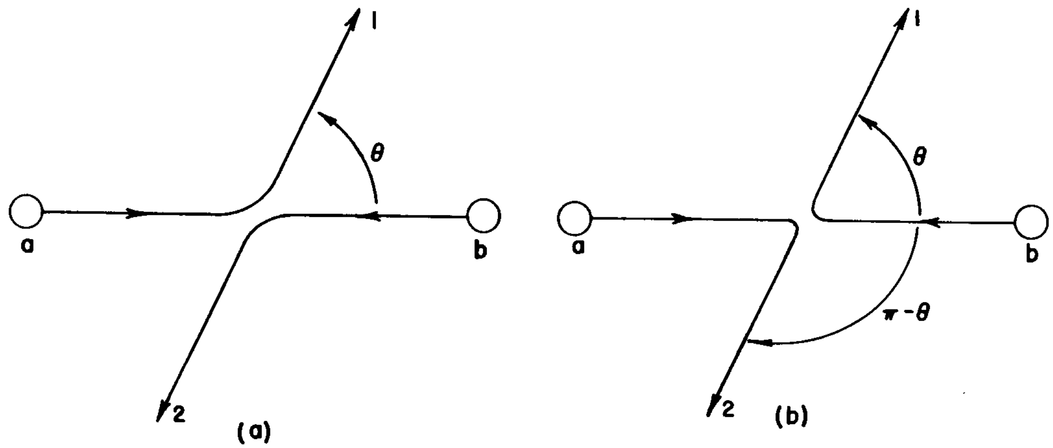
- Ordered: $$\sum_{i<j} |ij\rangle\langle ij| = 1$$
- Not ordered: $$\frac{1}{2!}\sum_{ij} |ij\rangle\langle ij| = 1$$

Bosons ordered: $$\sum_{i\leq j} |ij\rangle\langle ij| = 1$$

not ordered: $$\sum_{ij} \frac{n_1!n_2!\ldots}{2!} |ij\rangle\langle ij| = 1$$

Scattering of identical particles in their c of m

Particles that can be "distinguished"



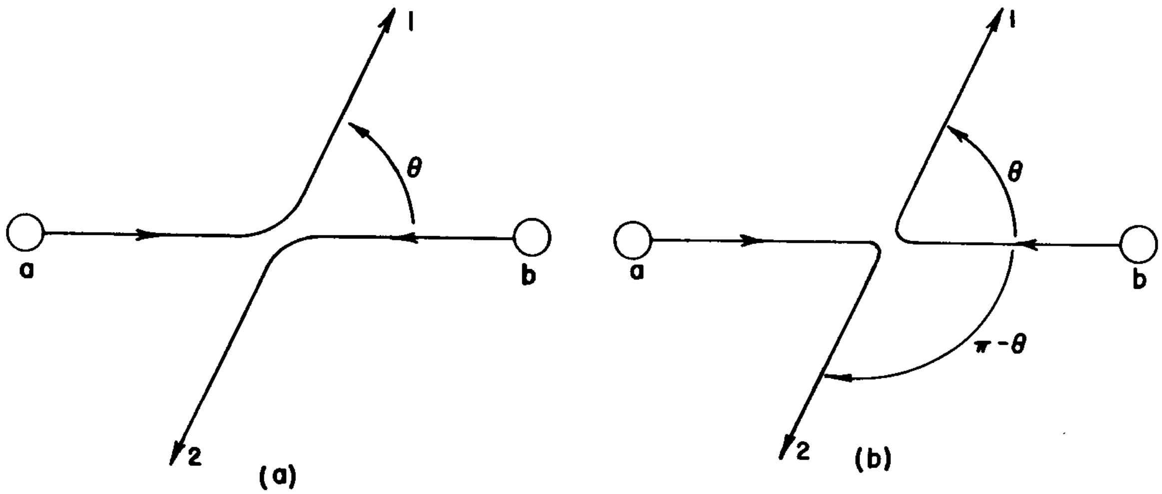
particle a in D1 (a) $$\frac{d\sigma}{d\Omega}(a\ in\ D_1, b\ in\ D_2) = |f(\theta)|^2$$

particle a in D2 (b) $$\frac{d\sigma}{d\Omega}(a\ in\ D_2, b\ in\ D_1) = |f(\pi - \theta)|^2$$

any particle in D1 $$\frac{d\sigma}{d\Omega}(particle\ in\ D_1) = |f(\theta)|^2 + |f(\pi - \theta)|^2$$

Identical bosons

- Cannot distinguish (a) and (b)



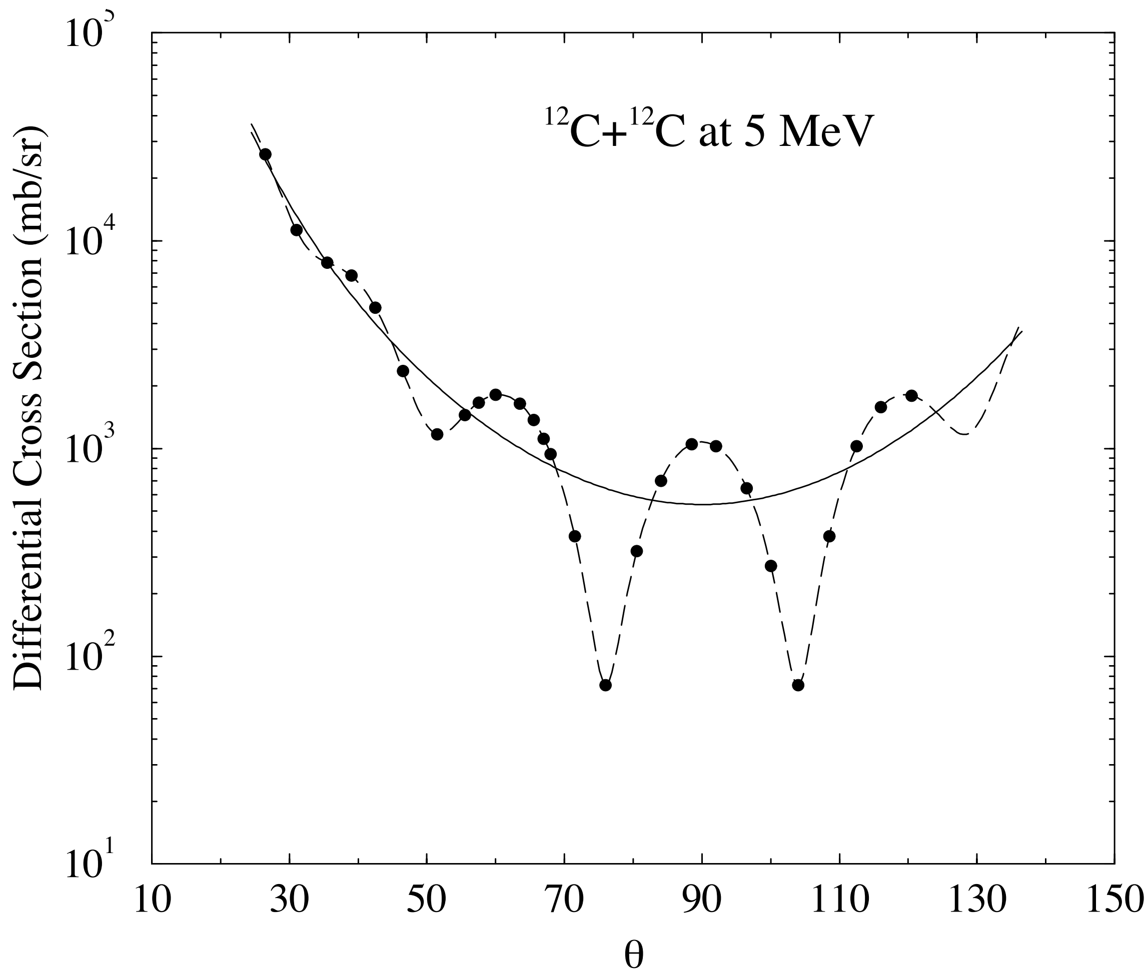
- Rule for bosons: add amplitudes then square!

$$\frac{d\sigma}{d\Omega}(bosons) = \ |f(\theta) + f(\pi - \theta)|^2$$

- Interference
- 90 degrees: factor of 2 compared to "classical" cross section

$^{12}C + {}^{12}C$

Low-energy boson-boson scattering

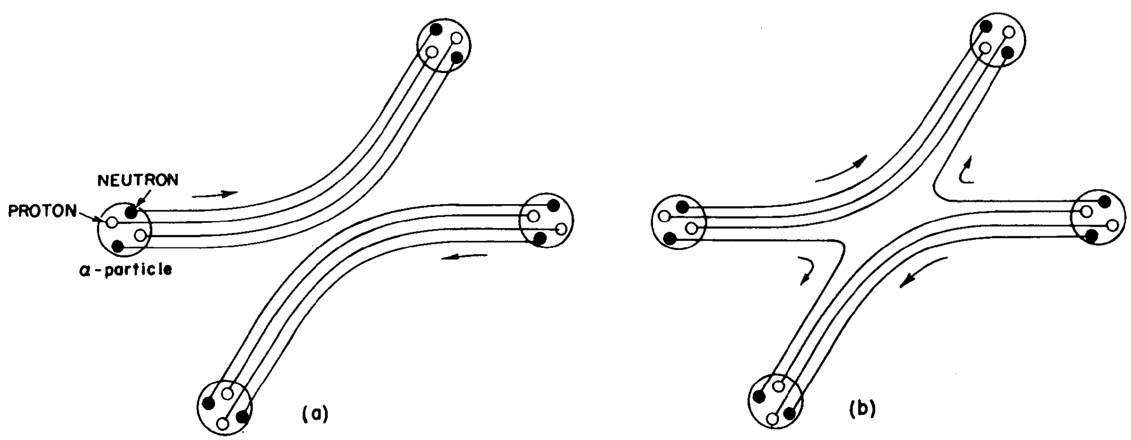


Phys. Rev. **123**, 878 (1961)

^{12}C a boson?

- 6 protons and 6 neutrons
- total angular momentum integer (made of 12 spin-$\frac{1}{2}$ particles)
- ground state 0^+
- first excited state above 4 MeV

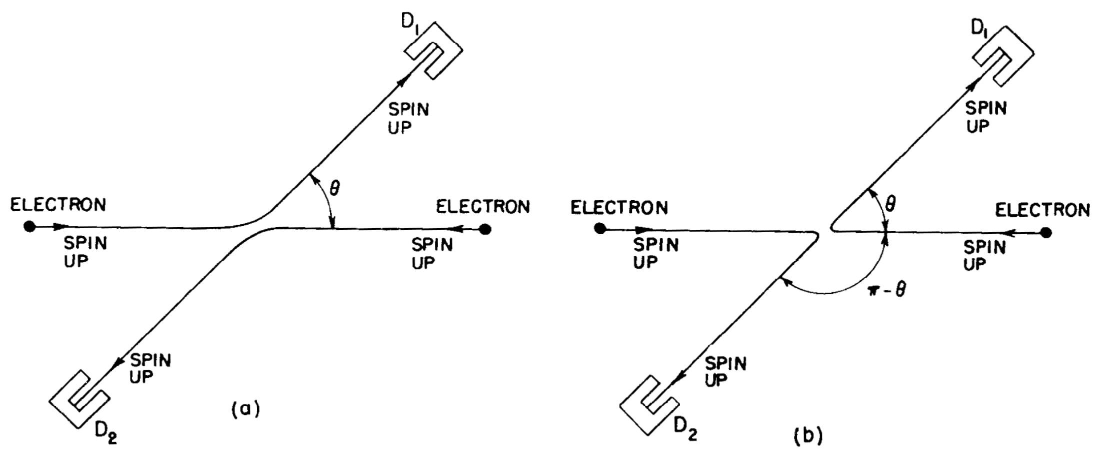
- ^{4}He atom: 2p + 2n + 2e $\Rightarrow$ boson
- ^{3}He atom: 2p + 1n + 2e $\Rightarrow$ fermion
- but



Fermion-fermion scattering

- Identical fermions: electrons with spin up

$$\frac{d\sigma}{d\Omega}(fermions) = \left|f(\theta) - f(\pi - \theta)\right|^2$$



- What about