Infinite systems & plane-wave states

- Suppress for now discrete quantum numbers (for fermions)
- Momentum eigenstates of kinetic energy

$$\frac{\boldsymbol{p}_{op}^2}{2m}\,|\boldsymbol{p}'\rangle = \frac{\boldsymbol{p}'^2}{2m}\,|\boldsymbol{p}'\rangle$$

- Associated wave function $$\langle \boldsymbol{r}|\boldsymbol{p}\rangle = \frac{1}{(2\pi\hbar)^{3/2}}e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{r}}$$
- Normalization condition $$\langle \boldsymbol{p}'|\boldsymbol{p}\rangle = \frac{1}{(2\pi\hbar)^3}\int d\boldsymbol{r}\; e^{\frac{i}{\hbar}(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{r}} = \delta(\boldsymbol{p}'-\boldsymbol{p})$$
- Often used: wave vectors $$\boldsymbol{k} = \frac{\boldsymbol{p}}{\hbar}$$
- Wave function $$\langle \boldsymbol{r}|\boldsymbol{k}\rangle = \frac{1}{(2\pi)^{3/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$
- and $$\langle \boldsymbol{k}'|\boldsymbol{k}\rangle = \delta(\boldsymbol{k}'-\boldsymbol{k})$$

Box normalization

- Confinement to cubic box $V = L^3$
- Wave function $\langle \boldsymbol{r} | \boldsymbol{k} \rangle = \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$
- Boundary conditions: only discrete $\langle \boldsymbol{k}' | \boldsymbol{k} \rangle = \delta_{\boldsymbol{k}',\boldsymbol{k}}$
- Means $\langle \boldsymbol{k}' | \boldsymbol{k} \rangle = \int_{box} d\boldsymbol{r}\ \langle \boldsymbol{k}' | \boldsymbol{r} \rangle \langle \boldsymbol{r} | \boldsymbol{k} \rangle = \frac{1}{V} \int_{box} d\boldsymbol{r}\ e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \delta_{\boldsymbol{k}',\boldsymbol{k}}$
- For example: periodic bc
- x-direction $e^{ik_x x} = e^{ik_x(x+L)} = e^{ik_x x} e^{ik_x L}$
- therefore $\cos(k_x L) + i \sin(k_x L) = 1$
- Hence $k_x = n_x \frac{2\pi}{L}$ where $n_x = 0, \pm 1, \pm 2, ...$
- Also for y and z
- Each allowed triplet $\{k_x, k_y, k_z\}$ corresponds to $\{n_x, n_y, n_z\}$
- Ground state: fill the lowest-energy states up to a maximum
- Fermi momentum; wave vector $p_F = \hbar k_F$

Thermodynamic limit

- Determine Fermi wave vector by calculating the expectation value of the number operator in the ground state

$$|\Phi_0\rangle = \prod_{|\boldsymbol{k}|<k_F,\mu} a^\dagger_{\boldsymbol{k}\mu} |0\rangle$$

- with μ representing discrete quantum numbers (spin, isospin)
- Thermodynamic limit $\quad N \to \infty$

$$V \to \infty$$

- with fixed density $\quad \rho = \dfrac{N}{V}$
- Replace summations by integrations for any function f

$$\sum_{\boldsymbol{k}\mu} f(\boldsymbol{k},\mu) = \sum_{n_x n_y n_z} \sum_{\mu} f(\frac{2\pi \boldsymbol{n}}{L},\mu)$$

$$L \to \infty \quad \Rightarrow \int d\boldsymbol{n} \sum_{\mu} f(\frac{2\pi \boldsymbol{n}}{L},\mu) = \frac{V}{(2\pi)^3} \int d\boldsymbol{k} \sum_{\mu} f(\boldsymbol{k},\mu)$$

Properties of Fermi gas ground state

- Remember $$N = \langle\Phi_0|\hat{N}|\Phi_0\rangle = \sum_{\boldsymbol{k}\mu}\langle\Phi_0|a^\dagger_{\boldsymbol{k}\mu}a_{\boldsymbol{k}\mu}|\Phi_0\rangle = \sum_{\boldsymbol{k}\mu}\theta(k_F-k)$$
$$= \frac{V}{(2\pi)^3}\sum_\mu\int d^3k\ \theta(k_F-k) = \frac{\nu V}{6\pi^2}k_F^3$$
- degeneracy ν so $k_F = \left\{\frac{6\pi^2 N}{\nu V}\right\}^{1/3}$ fixed ρ: k_F smaller if ν larger
- Energy from $$\hat{T} = \sum_{\boldsymbol{k}\mu}\sum_{\boldsymbol{k}'\mu'}\langle\boldsymbol{k}\mu|\frac{\hbar^2\boldsymbol{k}^2}{2m}|\boldsymbol{k}'\mu'\rangle a^\dagger_{\boldsymbol{k}\mu}a_{\boldsymbol{k}'\mu'} = \sum_{\boldsymbol{k}'\mu'}\frac{\hbar^2\boldsymbol{k}'^2}{2m}a^\dagger_{\boldsymbol{k}'\mu'}a_{\boldsymbol{k}'\mu'}$$
- yielding $$\hat{T}|\Phi_0\rangle = \left(\sum_{\boldsymbol{k}'\mu'}\frac{\hbar^2\boldsymbol{k}'^2}{2m}a^\dagger_{\boldsymbol{k}'\mu'}a_{\boldsymbol{k}'\mu'}\right)\prod_{|\boldsymbol{k}|<k_F\,\mu}a^\dagger_{\boldsymbol{k}\mu}|0\rangle$$
$$\hat{T}|\Phi_0\rangle = E_0|\Phi_0\rangle = \left(\sum_{|\boldsymbol{k}|<k_F,\mu}\frac{\hbar^2\boldsymbol{k}^2}{2m}\right)|\Phi_0\rangle$$
- and therefore $$E_0 = \sum_{|\boldsymbol{k}|<k_F,\mu}\frac{\hbar^2\boldsymbol{k}^2}{2m} = \frac{V}{(2\pi)^3}\sum_\mu\int d^3k\frac{\hbar^2k^2}{2m}\theta(k_F-k)$$
$$= V\frac{\nu}{(2\pi)^3}\,4\pi\,\frac{\hbar^2}{2m}\,\frac{1}{5}k_F^5$$
- written as $$\frac{E_0}{N} = \frac{V}{N}\,\frac{\nu}{2\pi^2}\,\frac{\hbar^2k_F^5}{10m} = \frac{3}{5}\,\frac{\hbar^2k_F^2}{2m} = \frac{3}{5}\varepsilon_F = \frac{3}{5}k_BT_F$$

Nuclear matter

- Key quantities
  - Saturation density: 0.16 nucleons per fm^3 ⇒ $k_F = 1.33 \text{ fm}^{-1}$ $\nu = 4$

    interparticle spacing $r_0 \approx 1.14 \text{ fm}$
  - Energy per particle at saturation: ~-16 MeV
- Relation between V_{NN} (including possible V_{NNN}) and these quantities still debated
- Bethe contributed ~10 years of his scientific life to this problem
- No global consensus on precise mechanism of saturation
  - role of pions
  - role of three-body interaction
  - role of relativity if any
  - many phenomenological ways to represent saturation properties

Neutron matter

- Interior of neutron star