Slides Chapter 1-7 Dickhoff-Van Neck

- Preliminary material covered in slides of Chs. 1-5 assumed more or less familiar
- Green's function formulation of single-particle problem in Ch.6 slides useful preparation for general formulation
- Single-particle propagator in many-fermion system introduced in Ch.7 slides

Symmetric and antisymmetric states

When is quantum physics expected?

Consider the energy levels for a particle of mass m enclosed in a box with volume $V = L^3$

$$\varepsilon_{n_x,n_y,n_z} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2)$$ positive integers

Total number of states below energy E

$$\Omega(E) = \frac{\pi}{6}\left(\frac{8mL^2E}{h^2}\right)^{3/2} = \frac{\pi}{6}\left(\frac{8mE}{h^2}\right)^{3/2} V$$

"Quantumness" --> indistinguishability not important when

$$1 \gg Q \equiv \frac{N}{\Omega} = \frac{6}{\pi}\rho\left(\frac{h^2}{12mk_BT}\right)^{3/2}$$

Use $E = \frac{3}{2}k_BT$

Q

System	T (K)	Density (m^{-3})	Mass (u)	Q
He (l)	4.2	1.9×10^{28}	4.0	1.1
He (g)	4.2	2.5×10^{27}	4.0	1.4×10^{-1}
He (g)	273	2.7×10^{25}	4.0	2.9×10^{-6}
Ne (l)	27.1	3.6×10^{28}	20.2	1.1×10^{-2}
Ne (g)	273	2.7×10^{25}	20.2	2.5×10^{-7}
e^- Na metal	273	2.5×10^{28}	5.5×10^{-4}	1.7×10^{3}
e^- Al metal	273	1.8×10^{29}	5.5×10^{-4}	1.2×10^{4}
e^- white dwarfs	10^{7}	10^{36}	5.5×10^{-4}	8.5×10^{3}
p,n nuclear matter	10^{10}	1.7×10^{44}	1.0	6.5×10^{2}
n neutron star	10^{8}	4.0×10^{44}	1.0	1.5×10^{6}
^{87}Rb condensate	10^{-7}	10^{19}	87	1.5

Bosons and Fermions

- Use experimental observations to conclude about consequences of identical particles
- Two possibilities
  - antisymmetric states ⇒ **fermions** half-integer spin
    - Pauli from properties of electrons in atoms
  - symmetric states ⇒ **bosons** integer spin
    - Considerations related to electromagnetic radiation (photons)
- Can also consider quantization of "field" equations
  - e.g. quantize "free" Maxwell equations (see standard textbooks)

Wolfgang Pauli (1900-1958)

- The Nobel Prize in Physics 1945 was awarded to Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle".

- paper Zeitschr. f. Phys. 31, 765 (1925)

Review single-particle states

- Notation $|\ldots\rangle$
- ... list of quantum numbers associated with a CSCO
- Normalization $\langle\alpha|\beta\rangle = \delta_{\alpha,\beta}$
- Continuous quantum numbers
  - Example $\langle \boldsymbol{r}, m_s|\boldsymbol{r}', m_s'\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}')\delta_{m_s,m_s'}$
- Completeness $\sum_\alpha |\alpha\rangle\langle\alpha| = 1$

Consequences for two-particle states

- CVS for two particles: product space
- Notation $|\alpha_1\alpha_2) = |\alpha_1\rangle|\alpha_2\rangle$
- Orthogonality $(\alpha_1\alpha_2|\alpha_1'\alpha_2') = \delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}$
- Completeness $\sum_{\alpha_1\alpha_2} |\alpha_1\alpha_2)(\alpha_1\alpha_2| = 1$

Exchange degeneracy

- Consider $\alpha_1 \neq \alpha_2$
- Then $|\alpha_2\alpha_1) \neq |\alpha_1\alpha_2)$
- All states $|\alpha_1\alpha_2)$

  $|\alpha_2\alpha_1)$

  $c_1|\alpha_1\alpha_2) + c_2|\alpha_2\alpha_1)$

  yield α_1 for one particle and α_2 for the other upon measurement
- Yet, unclear which state describes this system and therefore inconsistent with quantum postulates
- Consider permutation operator

$$P_{12}|\alpha_1\alpha_2) = |\alpha_2\alpha_1)$$

  with $P_{12} = P_{21}$ and $P_{12}^2 = 1$
- Hamiltonian for two particles is symmetric for 1 ⇔ 2

Development

- Typical Hamiltonian $H = \frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} + V(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$
- Consider operator acting on particle 1 and corresponding eigenvalue $A_1|\alpha_1\alpha_2) = a_1|\alpha_1\alpha_2)$
- Similarly, the same operator acting on particle 2 yields
$$A_2|\alpha_1\alpha_2) = a_2|\alpha_1\alpha_2)$$
- Note $P_{12}A_1|\alpha_1\alpha_2) = a_1 P_{12}|\alpha_1\alpha_2) = a_1|\alpha_2\alpha_1) = A_2|\alpha_2\alpha_1)$
- and $P_{12}A_1|\alpha_1\alpha_2) = P_{12}A_1P_{12}^{-1}P_{12}|\alpha_1\alpha_2) = P_{12}A_1P_{12}^{-1}|\alpha_2\alpha_1)$
- Holds for any state; therefore $P_{12}A_1P_{12}^{-1} = A_2$
- It follows that $P_{12}HP_{12}^{-1} = H$ or $[P_{12}, H] = 0$

Symmetric and antisymmetric two-particle states

- So $$[P_{12}, H] = 0$$
- Common eigenkets either

$$|\alpha_1\alpha_2\rangle_+ = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$$

or

$$|\alpha_1\alpha_2\rangle_- = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) - |\alpha_2\alpha_1)\}$$

- Eigenstates of the Hamiltonian either symmetric ⇒ bosons

or antisymmetric ⇒fermions

- Two-boson state $|\alpha_1\alpha_2\rangle_S = \left[\frac{1}{2n_\alpha!n_{\alpha'}!\ldots}\right]^{1/2}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$

$$\alpha_1 = \alpha_2 = \alpha \Rightarrow |n_\alpha = 2\rangle = |\alpha\alpha\rangle_S = |\alpha\rangle\,|\alpha\rangle$$

$$\alpha_1 \neq \alpha_2 \Rightarrow |\alpha_1\alpha_2\rangle_S = \frac{1}{\sqrt{2}}\{|\alpha_1\alpha_2) + |\alpha_2\alpha_1)\}$$

Fermions

- Antisymmetry: $|\alpha_2\alpha_1\rangle = -|\alpha_1\alpha_2\rangle$
- Both kets represent the same physical state: count only once in completeness relation ⇒ "order" quantum numbers

$$|1\rangle, |2\rangle, |3\rangle, \ldots$$

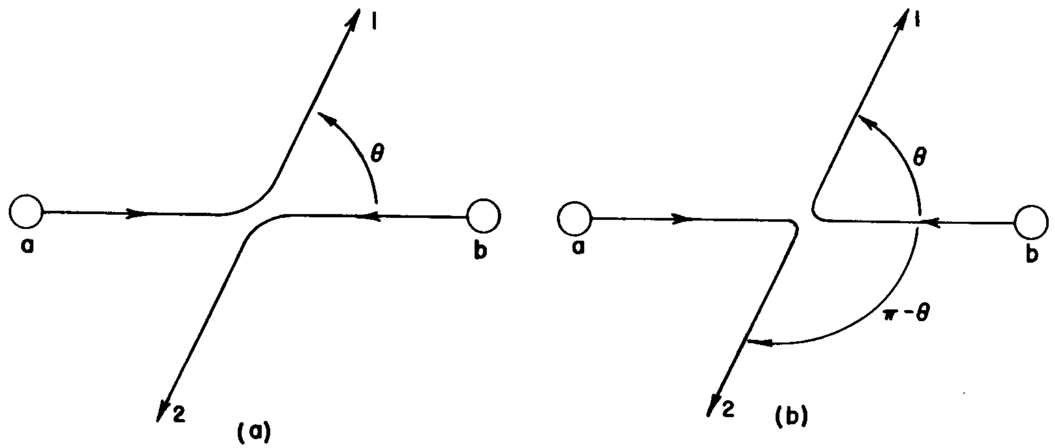
- Ordered: $\sum_{i<j} |ij\rangle\langle ij| = 1$
- Not ordered: $\frac{1}{2!}\sum_{ij} |ij\rangle\langle ij| = 1$

Bosons ordered: $\sum_{i\leq j} |ij\rangle\langle ij| = 1$

not ordered: $\sum_{ij} \frac{n_1!n_2!\ldots}{2!} |ij\rangle\langle ij| = 1$

Scattering of identical particles

Particles that can be "distinguished"



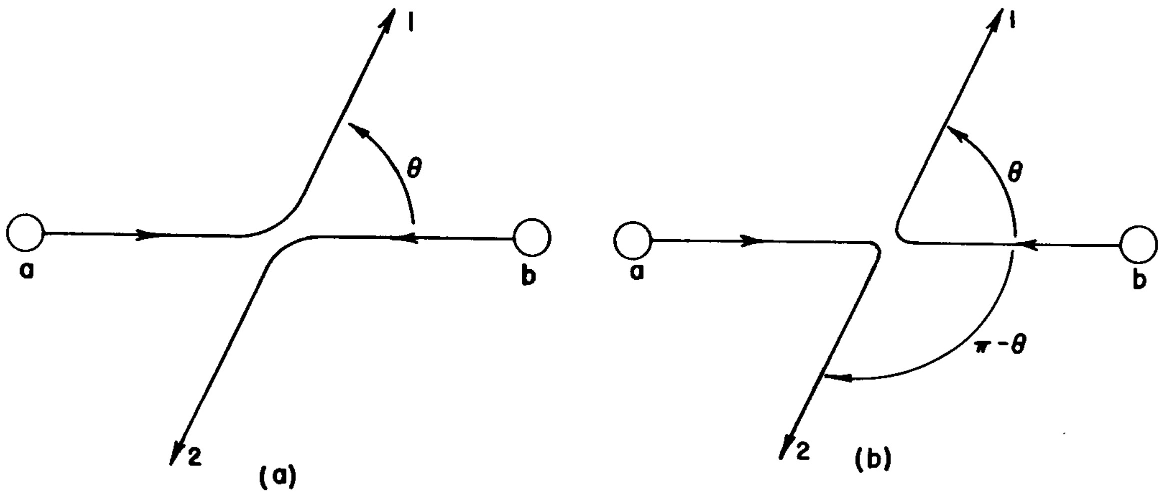
particle a in D1 (a) $$\frac{d\sigma}{d\Omega}(a\ in\ D_1, b\ in\ D_2) = \ |f(\theta)|^2$$

particle a in D2 (b) $$\frac{d\sigma}{d\Omega}(a\ in\ D_2, b\ in\ D_1) = \ |f(\pi - \theta)|^2$$

any particle in D1 $$\frac{d\sigma}{d\Omega}(particle\ in\ D_1) = \ |f(\theta)|^2 + |f(\pi - \theta)|^2$$

Identical bosons

- Cannot distinguish (a) and (b)



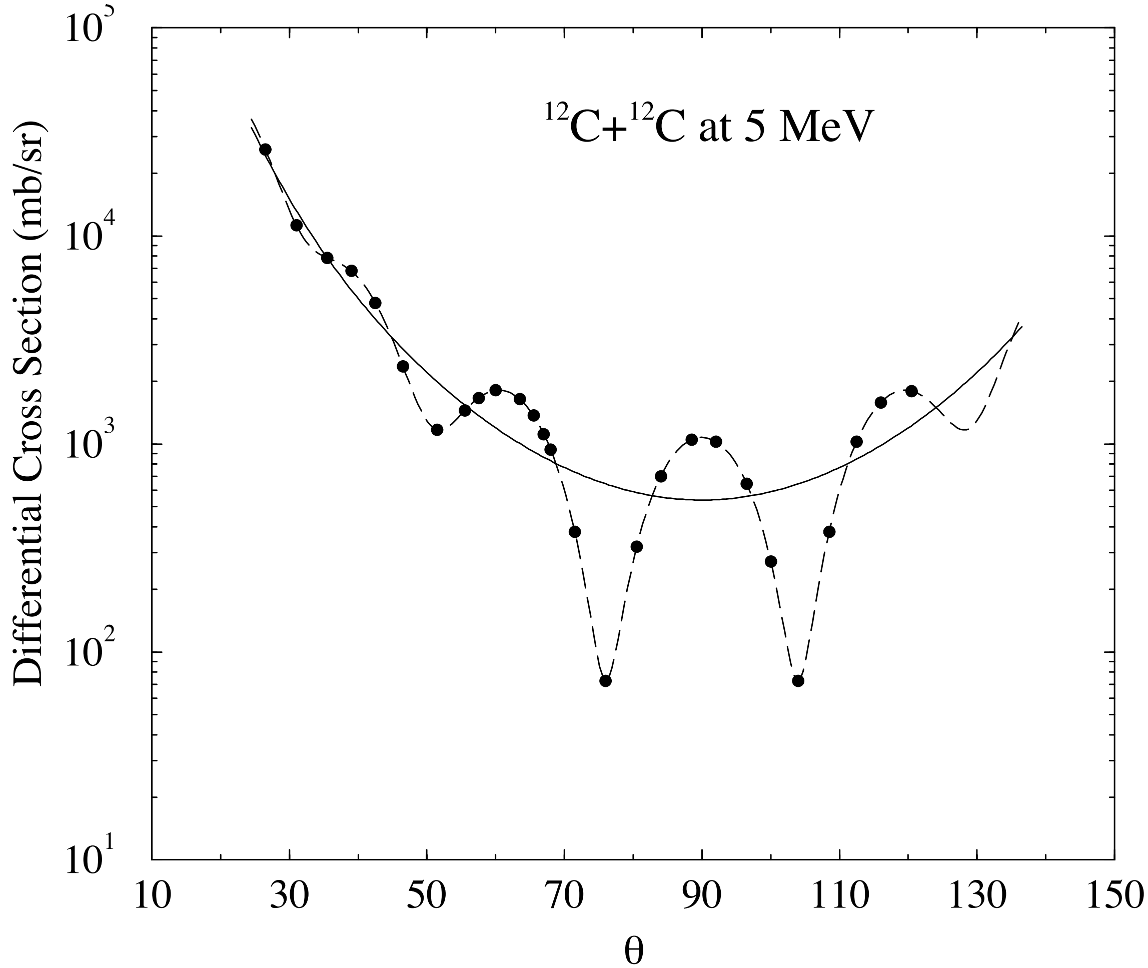
- Rule for bosons: add amplitudes then square!

$$\frac{d\sigma}{d\Omega}(bosons) = \left|f(\theta) + f(\pi - \theta)\right|^2$$

- Interference
- 90 degrees: factor of 2 compared to "classical" cross section

$^{12}C + {}^{12}C$

Low-energy boson-boson scattering

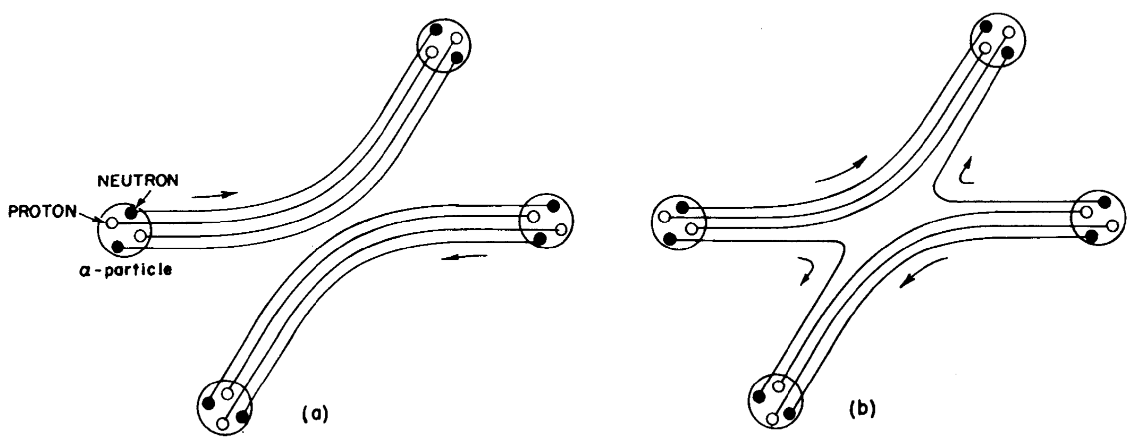


Phys. Rev. **123**, 878 (1961)

^{12}C a boson?

- 6 protons and 6 neutrons
- total angular momentum integer (made of 12 spin-$\frac{1}{2}$ particles)
- ground state 0^+
- first excited state above 4 MeV

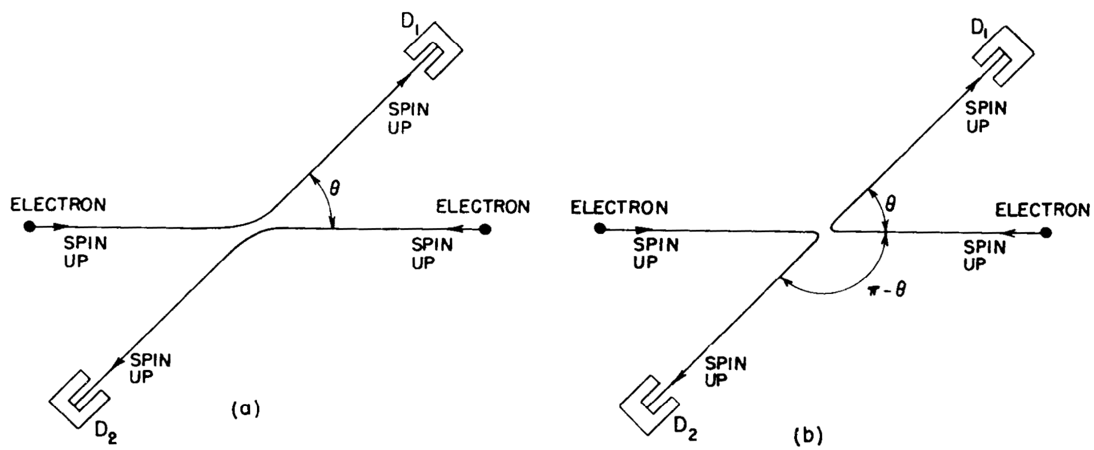
- ^{4}He atom: 2p + 2n + 2e ⇒**boson**
- ^{3}He atom: 2p + 1n + 2e ⇒**fermion**
- but



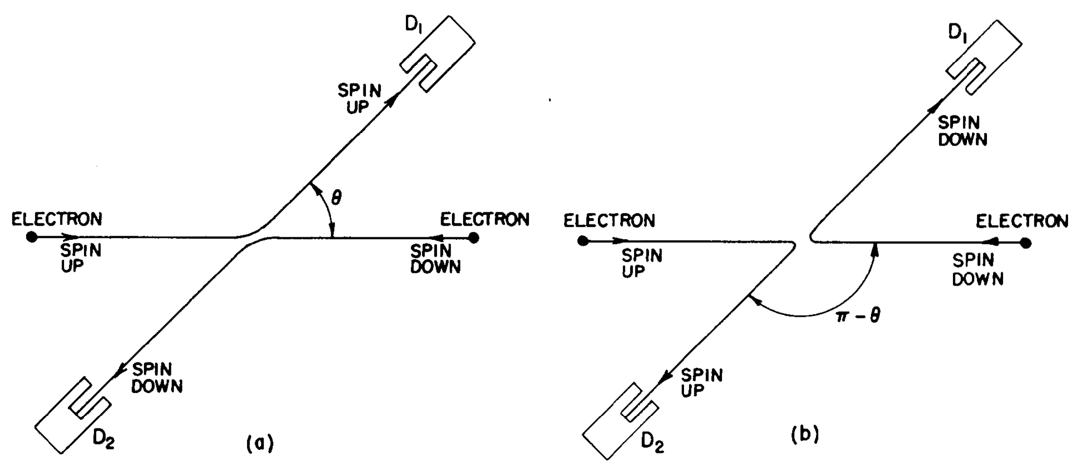
Fermion-fermion scattering

- Identical fermions: electrons with spin up

$$\frac{d\sigma}{d\Omega}(fermions) = \ |f(\theta) - f(\pi - \theta)|^2$$



- What about



N-particle states (fermions)

- Product states $|\alpha_1\alpha_2...\alpha_N) = |\alpha_1\rangle\,|\alpha_2\rangle\,...\,|\alpha_N\rangle$
- Normalization

$$\begin{aligned}(\alpha_1\alpha_2...\alpha_N|\alpha'_1\alpha'_2...\alpha'_N) &= \langle\alpha_1|\alpha'_1\rangle\langle\alpha_2|\alpha'_2\rangle...\langle\alpha_N|\alpha'_N\rangle \\ &= \delta_{\alpha_1,\alpha'_1}\delta_{\alpha_2,\alpha'_2}...\delta_{\alpha_N,\alpha'_N}\end{aligned}$$

- Completeness $\sum_{\alpha_1\alpha_2...\alpha_N} |\alpha_1\alpha_2...\alpha_N)(\alpha_1\alpha_2...\alpha_N| = 1$
- Identical particles: symmetric or antisymmetric states
- Fermions: use antisymmetrizer $\mathcal{A} = \frac{1}{N!}\sum_p (-1)^p P$
- Permutation operator: product of two-particle permutations
- # of two-particle permutations odd/even ⇒ **sign**

Example for 3 particles

- Check odd/even permutation

$$
\begin{aligned}
|\alpha_1\alpha_2\alpha_3\rangle &= \frac{1}{\sqrt{6}}\{|\alpha_1\alpha_2\alpha_3) - |\alpha_2\alpha_1\alpha_3) + |\alpha_2\alpha_3\alpha_1) \\
&- |\alpha_3\alpha_2\alpha_1) + |\alpha_3\alpha_1\alpha_2) - |\alpha_1\alpha_3\alpha_2)\}.
\end{aligned}
$$

- Note normalization (6 states)
- Also note antisymmetry $|\alpha_1\alpha_2\alpha_3\rangle = -|\alpha_2\alpha_1\alpha_3\rangle$
- No two fermions can occupy the same state!!
- Example for three bosons (symmetric state) [Check!]

$$
\begin{aligned}
|\alpha_1\alpha_1\alpha_2\rangle &= \frac{1}{\sqrt{3!2!}}\{|\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_2\alpha_1) \\
&+ |\alpha_2\alpha_1\alpha_1) + |\alpha_2\alpha_1\alpha_1) + |\alpha_1\alpha_2\alpha_1)\} \\
&= \frac{1}{\sqrt{3}}\{|\alpha_1\alpha_1\alpha_2) + |\alpha_1\alpha_2\alpha_1) + |\alpha_2\alpha_1\alpha_1)\}.
\end{aligned}
$$

N fermions

- Completeness with ordered single-particle (sp) quantum numbers

$$\sum_{\alpha_1\alpha_2\ldots\alpha_N}^{ordered} |\alpha_1\alpha_2\ldots\alpha_N\rangle\langle\alpha_1\alpha_2\ldots\alpha_N| = 1$$

- Not ordered $$\frac{1}{N!}\sum_{\alpha_1\alpha_2\ldots\alpha_N} |\alpha_1\alpha_2\ldots\alpha_N\rangle\langle\alpha_1\alpha_2\ldots\alpha_N| = 1$$

- Normalization with ordered single-particle (sp) quantum numbers

$$\begin{aligned}\langle\alpha_1\alpha_2\ldots\alpha_N|\alpha_1'\alpha_2'\ldots\alpha_N'\rangle &= \langle\alpha_1|\alpha_1'\rangle\langle\alpha_2|\alpha_2'\rangle\ldots\langle\alpha_N|\alpha_N'\rangle \\ &= \delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}\ldots\delta_{\alpha_N,\alpha_N'}\end{aligned}$$

- Not ordered ⇒ determinant

$$\langle\alpha_1\alpha_2\ldots\alpha_N|\alpha_1'\alpha_2'\ldots\alpha_N'\rangle = \begin{vmatrix} \langle\alpha_1|\alpha_1'\rangle & \langle\alpha_1|\alpha_2'\rangle & \ldots & \langle\alpha_1|\alpha_N'\rangle \\ \langle\alpha_2|\alpha_1'\rangle & \langle\alpha_2|\alpha_2'\rangle & \ldots & \langle\alpha_2|\alpha_N'\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\alpha_N|\alpha_1'\rangle & \langle\alpha_N|\alpha_2'\rangle & \ldots & \langle\alpha_N|\alpha_N'\rangle \end{vmatrix}.$$

Normalized N-particle wave function

- Called a Slater determinant

$$\psi_{\alpha_1\alpha_2\ldots\alpha_N}(x_1 x_2 \ldots x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \langle x_1|\alpha_1\rangle & \ldots & \langle x_N|\alpha_1\rangle \\ \langle x_1|\alpha_2\rangle & \ldots & \langle x_N|\alpha_2\rangle \\ \vdots & \ddots & \vdots \\ \langle x_1|\alpha_N\rangle & \ldots & \langle x_N|\alpha_N\rangle \end{vmatrix}.$$

- Hard to work with Slater determinants
- Use occupation number representation or second quantization